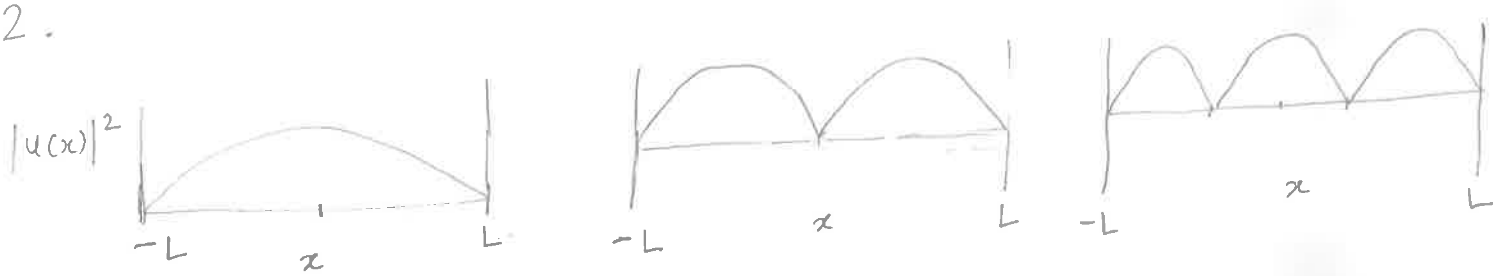
1 (a) $$\int_{-\infty}^{\infty} \hat{O}_2 \varphi \, \hat{O}_1^{\dagger} \psi^* \, dx = \langle \hat{O}_1 \psi | \hat{O}_2 \varphi \rangle$$

(b) $$\int_{-\infty}^{\infty} \hat{O}_1^{\dagger} \varphi^* \psi \, dx = \langle \hat{O}_1 \varphi | \psi \rangle$$

(c) $$\int_{-\infty}^{\infty} \varphi^* \varphi \, dx = \langle \varphi | \varphi \rangle$$

2.



3. $\hat{p}_x = -i\hbar \frac{\partial}{\partial x}$, $\hat{x} = x$ in position space representation

$$\langle \psi | \hat{p}_x \hat{x} - \hat{x} \hat{p}_x | \psi \rangle = \int_{-\infty}^{\infty} \psi^* (-i\hbar) \frac{\partial}{\partial x} (x\psi) \, dx - \int_{-\infty}^{\infty} \psi^* x (-i\hbar) \frac{\partial}{\partial x} \psi \, dx$$

$$= -i\hbar \left[\int_{-\infty}^{\infty} \psi^* (\psi + x\psi') \, dx - \int_{-\infty}^{\infty} \psi^* x \psi' \, dx \right]$$

$$= -i\hbar \int_{-\infty}^{\infty} \psi^* \psi \, dx = -i\hbar \cdot 1 = -i\hbar \qquad \therefore \; [\hat{p}_x, \hat{x}] = -i\hbar$$

4. $\psi(x) = A_0 \, e^{ikx} \, e^{-\alpha x}$, $0 \le x \le \infty$.

Then, $\psi^*(x) = A_0^* \, e^{-ikx} \, e^{-\alpha x}$

$$\langle p \rangle = \int_0^{\infty} (-i\hbar) \psi^* \frac{\partial \psi}{\partial x} \, dx = -i\hbar A_0 A_0^* \int_0^{\infty} e^{-ikx} e^{-\alpha x} \left[(-\alpha + ik) e^{-\alpha x + ikx} \right] dx$$

$$\langle p \rangle = -i\hbar |A_0|^2 \int_0^{\infty} e^{-\alpha x} (-\alpha + ik) e^{-\alpha x} \, dx$$

$$\langle p \rangle = -i\hbar |A_0|^2 (\alpha - ik) \int_0^\infty e^{-2\alpha x} dx$$

$$= \frac{i\hbar |A_0|^2 (\alpha - ik)}{2\alpha}$$

$$\langle p \rangle = \frac{|A_0|^2}{2\alpha} (k + i\alpha)\hbar.$$

$\hat{p}$ is a Hermitian operator. $\therefore$ $\langle p \rangle$ is expected to be a real number. In this problem, $\langle p \rangle$ is a complex number. The reason is that $\psi(x)$ is not a admissible state in Hilbert space. This is because in $0 \le x \le \infty$, $\psi(x)$ does not go to zero. Especially at $x = 0$, $\psi(x) \neq 0$. Hence, $\langle p \rangle$ becomes a complex number.

5. We are given that $\hat{H}\varphi_1 = E_1 \varphi_1$ and $\hat{H}\varphi_2 = E_2 \varphi_2$.

(i)
$$\Psi(x,T) = e^{-i\hat{H}T/\hbar} \Psi(x,0)$$

$$= e^{-i\hat{H}T/\hbar} \left[\frac{1}{\sqrt{2}} e^{-i\alpha_1} \varphi_1(x) + \frac{a}{\sqrt{3}} e^{-i\alpha_2} \varphi_2(x) \right]$$

$$= \frac{1}{\sqrt{2}} e^{-i\alpha_1} e^{-i\hat{H}T/\hbar} \varphi_1(x) + \frac{a}{\sqrt{3}} e^{-i\alpha_2} e^{-i\hat{H}T/\hbar} \varphi_2(x)$$

$$\Psi(x,T) = \frac{1}{\sqrt{2}} e^{-i\alpha_1} e^{-iE_1T/\hbar} \varphi_1(x) + \frac{a}{\sqrt{3}} e^{-i\alpha_2} e^{-iE_2T/\hbar} \varphi_2(x)$$

(ii) Probability amplitude to be found in state φ_2 is $\langle \varphi_2 | \Psi(x,T) \rangle$

$$\langle \varphi_2 | \Psi(x,T) \rangle = \frac{1}{\sqrt{2}} e^{-i\alpha_1} e^{-iE_1T/\hbar} \langle \varphi_2 | \varphi_1 \rangle + \frac{a}{\sqrt{3}} e^{-i\alpha_2} e^{-iE_2T/\hbar} \langle \varphi_2 | \varphi_2 \rangle$$

The required probability $\left| \langle \varphi_2 | \Psi(x,T) \rangle \right|^2 = \frac{a^2}{3}$

Note that $\langle \varphi_2 | \varphi_1 \rangle = 0$ and $\langle \varphi_2 | \varphi_2 \rangle = 1$.