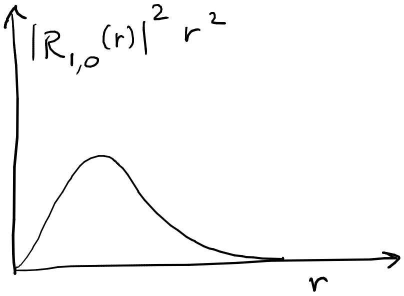
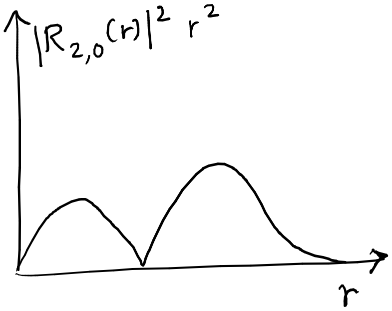
PHY 313 : Quiz 3 : Brief solution : Nov 2019

1 (a)





2. For a given quantum number N, there are

$l = 1, 2, 3 \dots N-1$ states.

For every l, $m = \underbrace{-l, -l+1, \dots\dots l-1, l}_{2l+1 \text{ states}}$.

Hence, degree of degeneracy is

$$\sum_{l=0}^{N-1} 2l+1 = 2\sum_{l=0}^{N-1} l + \sum_{l=0}^{N-1} 1$$

$$= 2\left[\frac{(N-1)N}{2}\right] + N = N^2$$

Two kinds of degeneracies are present.
For given l, $2l+1$ states are degenerate.
This arises due to rotational invariance of Hydrogen atom.

For a given N, states with different l are degenerate. This is called accidental degeneracy and arises due to conservation of Runge-Lenz vector.

3)

$$\psi_{N,N-1,m}(r,\theta,\varphi) \propto e^{-2r/Na_0}\, r^{N-1}\, Y_{N-1}^{m}(\theta,\varphi)$$

Probability in radial direction:

$$P(r)\, r^2 = |\psi_{N,N-1,m}|^2\, r^2 = e^{-\frac{2r}{Na_0}}\, r^{2N}$$

For $N=2$, $l=1$

$$\text{probability} = e^{-r/a_0}\, r^4$$

$$\frac{d}{dr}\left(e^{-r/a_0}\, r^4\right) = 0 \quad \text{gives} \quad r_0 = 4a_0$$

4) $$L_z \xrightarrow{(r,\theta,\varphi)\text{ basis}} -i\hbar\frac{\partial}{\partial\varphi}$$

$$-i\hbar\frac{\partial}{\partial\varphi}\psi_{l_z}(\varphi) = l_z\,\psi_{l_z}(\varphi)$$

Solving this, we get $\psi_{l_z}(\varphi) = \exp\left(i\frac{l_z}{\hbar}\varphi\right)$

Use boundary condition : $\psi(0) = \psi(2\pi)$

This gives $1 = e^{i2\pi l_z/\hbar} \Rightarrow \frac{l_z}{\hbar} = m \rightarrow$ integer

$\therefore\ l_z = m\hbar$

5) Note $J_x = \frac{J_+ + J_-}{2}$ and $J_y = \frac{J_+ - J_-}{2i}$

Then, $J_x - \frac{i}{2} J_y = \frac{1}{4}\left(J_+ + 3J_-\right) = \tilde{J}$

$$\langle j'm' | \tilde{J} | jm \rangle = \frac{\hbar}{4}\sqrt{(j-m)(j+m+1)}\ \delta_{jj'}\ \delta_{m+1,m'}$$
$$+ \frac{3\hbar}{4}\sqrt{(j+m)(j-m+1)}\ \delta_{jj'}\ \delta_{m-1,m'}$$

For $j=1$ subspace, $m = -1, 0, 1$

$\therefore$ $\tilde{J}$ matrix will have order 3.

Here's the matrix

$$\tilde{J} = \frac{\hbar}{\sqrt{2}} \begin{array}{cc} \begin{array}{ccc} (1,1) & (1,0) & (1,-1) \end{array} & \\ \begin{pmatrix} 0 & 1/2 & 0 \\ 3/2 & 0 & 1/2 \\ 0 & 3/2 & 0 \end{pmatrix} & \begin{array}{c} (1,1) \\ (1,0) \\ (1,-1) \end{array} \end{array}$$