PHY313 Test - 2 Brief solution

1) $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

Time evolution operator $e^{-i\hat{H}t/\hbar}$ and $\frac{\hat{H}}{\hbar\omega} = a^{+}a + \frac{1}{2}$

$$e^{-i\omega(a^{+}a+\frac{1}{2})} = e^{-i\omega/2}\, e^{-i\omega a^{+}a}$$

$$|\psi(t)\rangle = e^{-\hat{H}t/\hbar}|\psi(0)\rangle = \frac{e^{-i\omega/2}}{\sqrt{2}}\, e^{-i\omega t a^{+}a}\left(|0\rangle + |1\rangle\right)$$

Note $a^{+}a|n\rangle = n|n\rangle$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\, e^{-i\omega t/2}\left(|0\rangle + e^{-i\omega t}|1\rangle\right)$$

2) Problem done in class. Refer your notes.

3) $$U^{+}[R]\, P_y\, U[R] = \left(I + \frac{i\varepsilon}{\hbar}L_z\right) P_y \left(I - \frac{i\varepsilon}{\hbar}L_z\right)$$

$$= P_y - \frac{i\varepsilon}{\hbar}\left(P_y(xP_y - yP_x) - (xP_y - yP_x)P_y\right)$$

$$= P_y - \frac{i\varepsilon}{\hbar}\left(P_y x P_y - P_y y P_x - xP_yP_y + yP_xP_y\right)$$

Since $[x\, P_y] = 0$, $[y\, P_x] = 0$ and $[P_x\, P_y] = 0$

$$= P_y - \frac{i\varepsilon}{\hbar}\left(\cancel{P_y x P_y} - P_y y P_x - \cancel{P_y x P_y} + yP_yP_x\right)$$

$$= P_y - \frac{i\varepsilon}{\hbar}\left(-P_y\, y\, P_x + y\, P_y\, P_x\right)$$

$$= P_y - \frac{i\varepsilon}{\hbar}\underbrace{[y\ P_y]}_{\to\, i\hbar} P_x = P_y + \varepsilon P_x$$

Hence,

$$U^{\dagger}[R]\, P_y\, U[R] = P_y - \frac{i\varepsilon}{\hbar}[P_y\ L_z] = P_y + \varepsilon P_x$$

$$\therefore\ [P_y\ L_z] = i\hbar P_x$$

4(a) Bloch wave function $\varphi(x) = e^{ikx}\ u(x)$ is periodic if periodicity of e^{ikx} and $u(x)$ are rationally related.

$$\frac{d}{(2\pi/k)} = \frac{n}{N} \Rightarrow \frac{kd}{2\pi} = \frac{n}{N}$$

In this case, $kd = \left(\frac{2m+1}{4}\right) 2\pi$

periodicity : $\mathrm{LCM}(2m+1, 4)$

b) We need limit of $\alpha \to \infty$. In this limit, the quantisation condition will be consistent only if $\sin \mu a \to 0$.

That is, $\sin \mu a = 0 \Rightarrow \mu a = n\pi$ ($n \in$ integer)

$$\therefore\ \frac{\sqrt{2mE}}{\hbar}\, a = n\pi \Rightarrow E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$