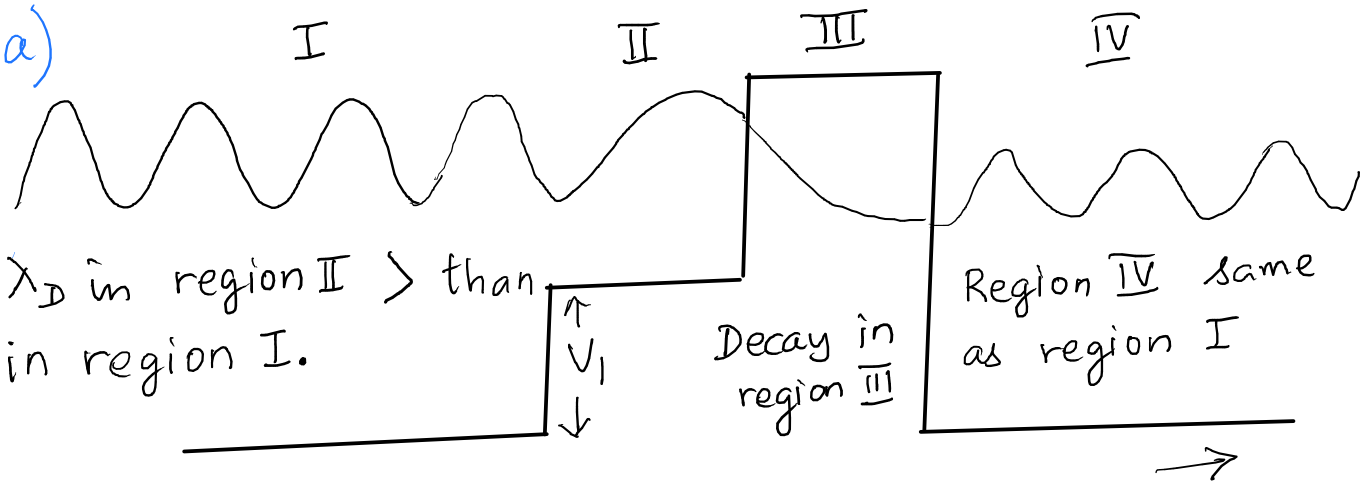
PHY313 • Midsem exam • Brief Solutions

1a)



b) Let the linear combination be $\alpha_1 u_1 + \alpha_2 u_2$.

Given that, $\hat{A}\, u_1 = a\, u_1$

$\hat{A}\, u_2 = a\, u_2$

$$\hat{A}(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 \hat{A} u_1 + \alpha_2 \hat{A} u_2$$
$$= \alpha_1 a u_1 + \alpha_2 a u_2 = a(\alpha_1 u_1 + \alpha_2 u_2)$$

c) $$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4}\langle \psi | [A,B]_+ | \psi \rangle^2 - \frac{1}{4}\langle \psi | [A,B] | \psi \rangle^2$$

We can choose $|\psi\rangle$ such that $\langle \psi | [A,B]_+ | \psi \rangle = 0$.

Hence, if $[A,B]=0$, then $(\Delta A)^2 (\Delta B)^2 \geq 0$.

The product of uncertainties can be 0.

2(a) $$\int_{-\infty}^{\infty} \psi^* \frac{d\psi}{dx}\, dx = \cancel{|\psi|^2}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \psi \frac{d\psi^*}{dx}\, dx$$

$\swarrow 0$

If $\hat{A} = \frac{d}{dx}$, then we have $\langle \psi | A \psi \rangle = -\langle A\psi | \psi \rangle$

Hence, $\hat{A}$ is not Hermitian.

It can be inferred that $A^{+} = -A$.

That is, $\left(\frac{d}{dx}\right)^{+} = -\frac{d}{dx}$.

b) $[\hat{p}, \hat{x} \sin \hat{p}] = [\hat{p}, \hat{x}] \sin \hat{p} - \hat{x} [\hat{p}, \sin \hat{p}]$

$$= -i\hbar \sin \hat{p} - 0$$

$$= -i\hbar \sin \hat{p}$$

c) We know, $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x}$

$[\hat{x}, \hat{p}]_{+} = \hat{x}\hat{p} + \hat{p}\hat{x}$

Then $\hat{x}\hat{p} = \frac{1}{2}\left([\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]_{+}\right)$

$$= \frac{i\hbar}{2} + \frac{1}{2} [\hat{x}, \hat{p}]_{+} \rightarrow \text{operator for } \hat{x}\hat{p}$$

In this, $[\hat{x}, \hat{p}]_{+}$ is not Hermitian. Hence, $\hat{x}\hat{p}$ is not Hermitian.

Note that $\langle \hat{x}\hat{p} \rangle = \frac{i\hbar}{2} + \frac{1}{2} \langle [\hat{x}, \hat{p}]_{+} \rangle$. Though operator exists, this quantity is not measurable since its expectation value is imaginary.

3(a) Using Erhenfest theorem for operator $\hat{A}$:

$$\frac{d}{dt} \langle \hat{A} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{A}] \rangle$$

If $\hat{A} = \hat{x}^2$, then

$$\frac{d}{dt}\langle x^2 \rangle = \frac{i}{\hbar}\langle [H, x^2] \rangle$$

Taking $\hat{H} = \frac{\hat{P}^2}{2m} + V(x)$, we get

$$\left[\frac{\hat{P}^2}{2m} + V(x), \hat{X}^2\right] = \left[\frac{\hat{P}^2}{2m}, \hat{X}^2\right] + \left[V(\hat{X}), \hat{X}^2\right]$$

$$= \frac{1}{2m}\left[\hat{P}^2, \hat{X}^2\right] + 0.$$

$$= \frac{1}{2m}\left(\left[\hat{P}^2, \hat{X}\right]\hat{X} + \hat{X}\left[P^2, X\right]\right)$$

Since $[P^2, X] = -2i\hbar P$

$$\left[\frac{P^2}{2m} + V(x), X^2\right] = \frac{-2i\hbar}{2m}\left(\hat{P}\hat{X} + \hat{X}\hat{P}\right)$$

$$\left[\hat{H}, \hat{X}^2\right] = \frac{\hbar}{im}\left(\hat{P}\hat{X} + \hat{X}\hat{P}\right)$$

$$\therefore \quad \frac{d}{dt}\langle x^2 \rangle = \frac{i}{\hbar}\left\langle \frac{\hbar}{im}\left(\hat{P}\hat{X} + \hat{X}\hat{P}\right)\right\rangle$$

$$m\frac{d\langle x^2 \rangle}{dt} = \langle \hat{P}\hat{X} + \hat{X}\hat{P} \rangle = \langle \hat{P}\hat{X} \rangle + \langle \hat{X}\hat{P} \rangle$$

(b) $\displaystyle\int_0^L A^2 |\psi(x,0)|^2 \, dx = 1,$ $\quad A \rightarrow$ normalisation constant

$$A^2 \int_0^L x^2 (x-L)^2 \, dx = 1$$

$$A^2 \frac{L^5}{30} = 1 \implies A = \sqrt{\frac{30}{L^5}}$$

Average energy $\langle E \rangle = \int_0^L \psi^* \left(\frac{-\hbar^2}{2m} \right) \frac{d^2\psi}{dx^2} \, dx$

$$\langle E \rangle = \frac{-\hbar^2}{2m} \int_0^L \psi^* \frac{d^2\psi}{dx^2} \, dx$$

$$\psi = A\, x(x-L), \qquad \psi'' = 2A.$$

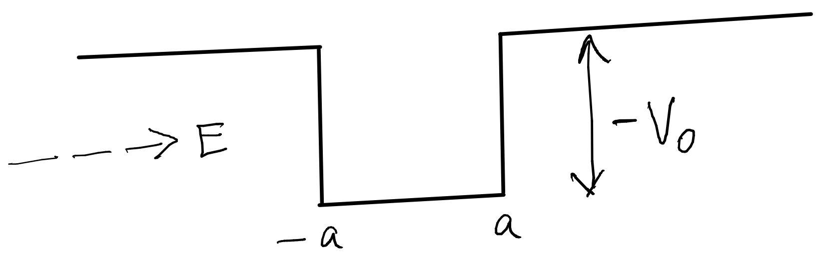
Then, $\langle E \rangle = -\frac{2\hbar^2 A^2}{2m} \int_0^L x(x-L) \, dx$

$$= -\frac{\hbar^2 A^2}{m} \left(-\frac{L^3}{6} \right) = \frac{\hbar^2 A^2 L^3}{3}$$

$$= \frac{\hbar^2 L^3}{6} \left(\frac{30}{L^5} \right) = \frac{5\hbar^2}{mL^2}$$

$$\langle E \rangle = \frac{5\hbar^2}{mL^2}$$

4(a) $V(x) = -V_0 \, \theta(a - |x|)$



Schrödinger equation

$(|x|>a)$ $$\frac{d^2\psi_1}{dx^2} - k^2\psi_1 = 0, \qquad k = \sqrt{\frac{-2mE}{\hbar}}$$

$(|x|<a)$ $$\frac{d^2\psi_2}{dx^2} + q^2\psi_2 = 0, \qquad q = \sqrt{\frac{2m(E+V_0)}{\hbar}}$$

Since we need only even states, inside the well $(|x|<a)$, our solution is

$$\psi_2(x) = A\cos qx.$$

For $|x|>a$, we expect decaying solutions.

$$\psi_1(x) = e^{kx} \quad (x<a)$$
$$= e^{-kx} \quad (x>a)$$

Boundary conditions are

$\psi_1(-a) = \psi_2(-a)$, and $\psi_1'(-a) = \psi_2'(-a)$

$$e^{-ka} = A\cos qa \qquad k e^{-ka} = Aq\sin qa$$

Dividing the two equations, we get

$$\frac{Aq\sin qa}{A\cos qa} = \frac{k e^{-ka}}{e^{-ka}}$$

$$q\tan qa = k$$

$$\Rightarrow \quad \tan qa = (k/q)$$

(b) This problem done in class.
Refer to your notes.

5 (a) Problem done in class. Refer to your notes.

(b) $$u(x,0) = c\left[3\,\psi_0(x) + 4\,\psi_1(x)\right]$$

$$\int_{-\infty}^{\infty} |u(x,0)|^2\, dx = 1$$

$$c^2 \int_{-\infty}^{\infty} 9|\psi_0|^2 + 16\,|\psi_1|^2 + \cancel{24\,\psi_0\psi_1}^{\,\to 0}\; dx$$

↳ orthogonal eigenstates

$$\therefore\; c^2(9+16) = 1 \implies c^2 = \frac{1}{25} \implies c = \pm\frac{1}{5}$$

$$u(x,t) = e^{-i\hat{H}t/\hbar}\, u(x,0)$$

$$= c\, e^{-i\hat{H}t/\hbar}\left(3\,\psi_0 + 4\,\psi_1\right)$$

$$= c\Big(\underbrace{3\, e^{-iE_0 t/\hbar}}_{C_0}\,\psi_0 + \underbrace{4\, e^{-iE_1 t/\hbar}}_{C_1}\,\psi_1\Big)$$

$$E_0 = \frac{\hbar\omega}{2} \quad \text{and} \quad E_1 = \frac{3}{2}\hbar\omega$$

$$\langle x\rangle = \int_{-\infty}^{\infty} u^*(x,t)\, \hat{x}\, u(x,t)\, dx$$

$$\langle x\rangle = c^2 \int_{-\infty}^{\infty} \left(c_0^* \psi_0 + c_1^* \psi_1\right) \hat{x} \left(c_0 \psi_0 + c_1 \psi_1\right) dx$$

$$\langle x\rangle = c^2 \int_{-\infty}^{\infty} \left[|c_0|^2 \psi_0^2 + |c_1|^2 \psi_1^2 + \psi_0 \psi_1 \left(c_0^* c_1 + c_0 c_1^*\right)\right] x \, dx$$

$\psi_0^2 x$ and $\psi_1^2 x$ are both odd functions.
They would vanish under the integral

$$\langle x\rangle = c^2 \int_{-\infty}^{\infty} x \, \psi_0 \, \psi_1 \; 2 \, \mathrm{Re}\left(c_0^* c_1\right) dx$$

$$\langle x\rangle = 2c^2 \, \mathrm{Re}\left(c_0^* c_1\right) \int_{-\infty}^{\infty} x \, \psi_0(x) \, \psi_1(x) \, dx$$

$$\mathrm{Re}\left(c_0^* c_1\right) = \mathrm{Re}\left(3 e^{iE_0 t/\hbar} \, 4 e^{-iE_1 t/\hbar}\right)$$

$$= 12 \, \mathrm{Re}\left(e^{i(E_0 - E_1)t/\hbar}\right)$$

$$= 12 \, \mathrm{Re}\left(e^{\frac{i}{\hbar}\left(\frac{\hbar\omega}{2} - \frac{3\hbar\omega}{2}\right)t}\right)$$

$$= 12 \, \mathrm{Re}\left(e^{-i\omega t}\right) = 12 \cos \omega t$$

Hence,

$$\langle x \rangle_t = 24\, c^2 \cos \omega t \; I = \frac{24\, I}{25} \cos \omega t$$

where $$I = \int_{-\infty}^{\infty} x\, \psi_0(x)\, \psi_1(x)\, dx$$

Note that I is simply a number.

The result above for $\langle x \rangle_t$ shows that expectation value oscillates with classical frequency ω of the oscillator.

Except for evaluating I, we need not know the explicit form of $\psi_0(x)$ and $\psi_1(x)$.