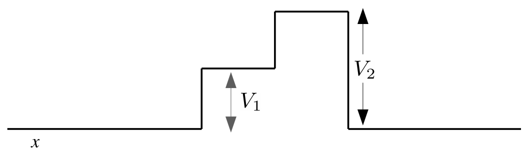
Indian Institute of Science Education and Research Pune
Mid-semester Exam, Aug (2019) semester.

Course name: Quantum Mechanics I	**Course code:** PHY 313
Date: 23.9.2019 (10:00 AM to 12:00 PM)	*Duration:* 2 hours
Instructor : M. S. Santhanam	Maximum marks: 50

- This question paper has 2 pages and 5 questions. All of them are compulsory.
- If you draw sketches as an answer, label the axes. No marks without axes labels.
- SHOW ALL THE STEPS CLEARLY in your calculations while arriving at an answer.
- Unless specified otherwise, all the symbols have their usual meanings.
- Use the same symbols and notation given in the question. Do not use your own.

1.(a) A barrier potential in one dimension is shown in the figure below. A particle incident from the left has definite energy E such that $V_1 < E < V_2$. First, draw the potential from this figure as it is and on top it, sketch the wavefunction for all x.



(b) An operator $\widehat{A}$ has two eigenstates u_1 and u_2 with same eigenvalue a. Show that any linear combination of u_1 and u_2 will have the same eigenvalue a.

(c) If $[\widehat{A}, \widehat{B}] = 0$ for any two Hermitian operators, what can be said about the uncertainty product $\Delta A \; \Delta B$. Explain your answer quantitatively. (3+3+4)

2.(a) Show that the operator $\frac{d}{dx}$ is anti-Hermitian and use it to infer the value of $(d/dx)^\dagger$.

(b) Find the value of the commutator $[\widehat{p}, \widehat{x} \; \sin(\widehat{p})]$.

(c) Consider a particle in one-dimensional potential. Show that the operator $\widehat{x}\widehat{p}$ is not Hermitian. We need the average of the observable $\langle xp \rangle$. Construct a Hermitian operator corresponding to this observable. Show that this operator is Hermitian. (You can use the known results to show that it is Hermitian). (3+3+4)

3(a) For Hermitian operators x and p, use the Ehrenfest theorem to show that

$$m \frac{d}{dx} \langle x^2 \rangle = \frac{1}{i\hbar} \left(\langle xp \rangle + \langle px \rangle \right)$$

(b) A particle is in a one-dimensinal box (infinite well) with walls at $x = 0$ and $x = L$. At $t = 0$, the state of the particle is

$$\psi(x, 0) = A \; x(x - L)$$

Find the value of normalisation constant A. What is the average energy at $t = 0$.

(6+4)

4(a) Consider the potential $V(x) = -V_0\ \Theta(a - |x|)$, with $V_0 > 0$ and Θ is the unit step function. If E is the energy of the particle lying in the range, $-V_0 \leq E \leq 0$, then show that the quantisation condition for even states is

$$\tan qa = \kappa/q,$$

where $\kappa = \sqrt{-2mE}/\hbar$ and $q = \sqrt{2m(E+V_0)}/\hbar$.

(b) Consider a particle in the potential $V(x) = -V_0\ \delta(x)$, with $V_0 > 0$. Show that

$$\psi'(0+) - \psi'(0-) = -\frac{2mV_0}{\hbar^2}\psi(0)$$

where $\psi'(0\pm)$ represents slope of the wavefunction upon approaching $x = 0$ from positive/negative side of the axis. Find $\psi(0)$.

(5+5)

5(a) For the *classical* harmonic oscillator (with energy E) given by,

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 x^2$$

Show that the classical probability to find the particle in the region between x and $x + dx$ is given by,

$$P(x)\ dx = \frac{1}{\pi}\frac{dx}{\sqrt{x_c^2 - x^2}}$$

(b) A particle is in a oscillator potential and has the initial state

$$u(x, 0) = C\,(3\psi_0(x) + 4\psi_1(x))$$

In this, $\psi_0(x)$ and $\psi_1(x)$ are harmonic oscillator eigenstates. Find the constant C. Find $\langle x\rangle$ at any other time t. Show that it oscillates with frequency ω. (5+5)

Some useful information:

Harmonic oscillator eigenstates :

$$u_n(x) = \left(\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$$

Recurrence relation for Hermite Polynomial :

$$H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}$$