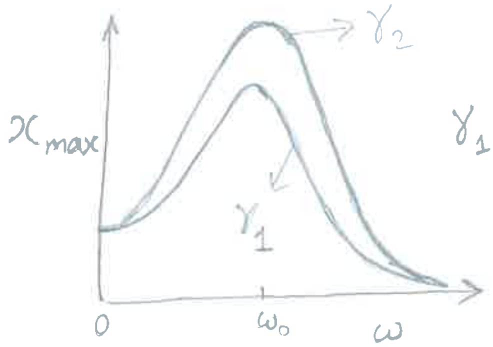
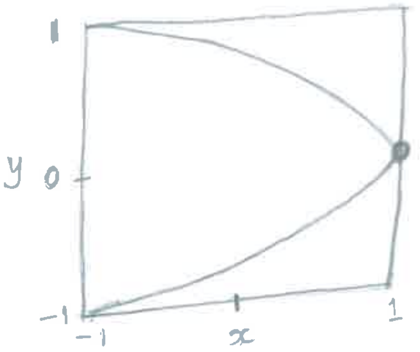
PHY 102 : Quiz 1 : Brief Solutions 4.2.2019

1)



2)



Let $\theta = \omega t$. Then,

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$x = 1 - 2y^2$$

$$\therefore \boxed{y^2 = \frac{1-x}{2}}$$

So, Lissajou's figure has parabolic shape.

Put $t=0$ to get initial point at $(1,0)$.

3) Dimension of mechanical impedance MT^{-1}.

4) Given $x(t) = A\left[\cos(\omega t+\alpha) - \sin(\omega t+\alpha)\right]$, $x(0)=0$, $v(0)=-1$.

(a) The conditions at $t=0$ give,

$$A(\cos\alpha - \sin\alpha) = 0 \quad \text{and} \quad -A\omega(\cos\alpha + \sin\alpha) = -1$$

Since $A \neq 0$, $\cos\alpha = \sin\alpha \implies \alpha = \pi/4$.

From 2nd equation, $A = \dfrac{1}{\omega(\cos\alpha + \sin\alpha)} = \dfrac{1}{\omega\sqrt{2}}$ (since $\alpha = \pi/4$).

(b) Total energy $E = \frac{1}{2}M\omega^2 x^2 + \frac{1}{2}M\dot{x}^2$

$$x(t) = \frac{1}{\sqrt{2}}\left[\cos(\omega t+\pi/4) - \sin(\omega t+\pi/4)\right], \quad \dot{x}(t) = \frac{-\omega}{\sqrt{2}}\left[\sin(\omega t+\pi/4) + \cos\left(\omega t+\frac{\pi}{4}\right)\right]$$

Substituting these in E and simplifying, we obtain

$$E = \frac{M}{2}$$

5) For damped harmonic motion, the solution is

$$x(t) = A\, e^{-\frac{\gamma t}{2m}} \sin(\omega' t + \varphi), \quad \text{where} \quad \omega' = \left(\frac{k}{m} - \frac{\gamma^2}{4m^2}\right)^{1/2}$$

We can write amplitude at any time t to be

$$A(t) = A_0\, e^{-\frac{\gamma t}{2m}}$$

Condition is $A(t) = \frac{1}{2} A_0$ at $t = \tau$.

$$\therefore \quad \frac{1}{2} = e^{-\frac{\gamma\tau}{2m}}$$

Taking $\ln$ both sides, we get

$$\ln \tfrac{1}{2} = -\frac{\gamma\tau}{2m} \Rightarrow \tau = -\frac{2m}{\gamma} \ln(1/2)$$

$$\therefore \quad \tau = \frac{2m}{\gamma} \ln 2$$

6) $\omega_0 = \sqrt{k/m}$ and $\omega_1 = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}$

$$\omega_0 - \omega_1 = \sqrt{\frac{k}{m}} - \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} = \sqrt{\frac{k}{m}}\left[1 - \left(1 - \frac{\gamma^2}{4m^2}\frac{m}{k}\right)^{1/2}\right]$$

In the limit $\gamma \Rightarrow 0$, $\left(1 - \frac{\gamma^2}{4m^2} \cdot \frac{m}{k}\right)^{1/2} \simeq 1 - \frac{\gamma^2}{8m^2} \cdot \frac{m}{k}$

(using Binomial expansion)

$$\therefore \quad \omega_0 - \omega_1 = \sqrt{\frac{k}{m}}\left(\frac{\gamma^2}{8m^2}\frac{m}{k}\right)$$

Use $\omega_0 = \sqrt{\frac{k}{m}}$ and $Q = \frac{m\omega_0}{\gamma}$.

Then, $\omega_0 - \omega_1 = \omega_0\left(\frac{1}{8Q^2}\right) \Rightarrow \frac{\omega_0 - \omega_1}{\omega_0} = \frac{1}{8Q^2}$

NOTE: For problem 2, another method is to put $\omega t = 0, \pi/4, \pi/2, \pi$ etc. Put the points (x, y) and join them to get the required **Lissajous** figure.