PHY102 - Mid Sem exam - Jan 2019 semester
Brief Solutions

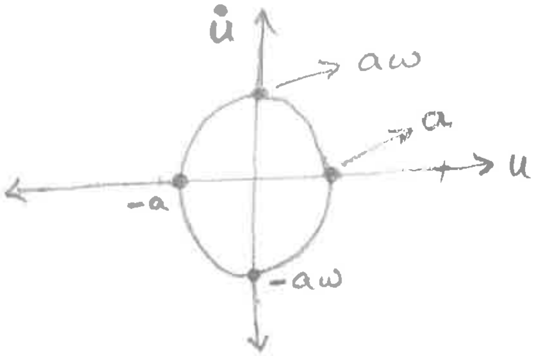
1 (a) At $t=0$, $x=0$. Then, $A\cos\alpha = 0$

$A \neq 0$. Then, $\cos\alpha = 0$. $\therefore \alpha = \pi/2$.

Given that time period $T = 2$ sec. $\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$

$\therefore \alpha = \pi/2$ and $\omega = \pi$.

(b)



$u(t) = a \sin\omega t$

$\dot{u}(t) = a\omega \cos\omega t$

Assume, $\omega > 1$.

(c) $x(t) = \sin 2\omega t$ and $y(t) = \sqrt{2}\, \sin(\omega t + \pi/4)$

$$= \sqrt{2}\left(\sin\omega t \cos\tfrac{\pi}{4} + \cos\omega t \sin\pi/4\right)$$

$$y(t) = \sin\omega t + \cos\omega t$$

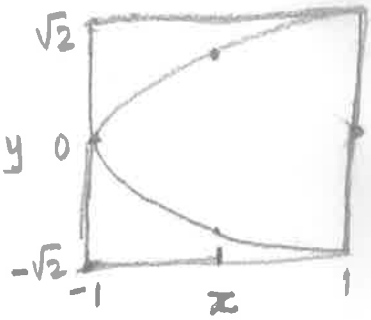
Now, $y^2 = (\sin\omega t - \cos\omega t)^2 = \sin^2\omega t + \cos^2\omega t + 2\sin\omega t \cos\omega t$

$$= 1 - \sin 2\omega t = 1 + x$$

$\therefore \boxed{y^2 = 1 - x}$

Since $-1 \le x \le 1$, put $x = -1, 0, 1$ and determine y.

Draw few points and join them together



This can also be drawn by choosing some simple values for t and plotting x vs. y.

2 (a) $x(t) = 3\sin(\pi t + \pi/4)$

If energy is purely kinetic, potential energy is zero.

$\therefore$ P.E $= \frac{1}{2} m\omega^2 x^2 = \frac{1}{2} m\omega^2\, 3^2 \sin^2(\pi t + \pi/4) = 0$

Since $m \neq 0$ and $\omega \neq 0$, $\sin^2(\pi t + \pi/4) = 0$.

$\Rightarrow$ $\sin(\pi t + \pi/4) = 0$ (or) $\pi t + \pi/4 = k\pi$, $k = 1, 2, 3 \ldots$

$$\therefore\ t = k - \frac{1}{4} = \frac{4k-1}{4}.$$

Note: k can be negative integers. Then, t would have -ve values. This would also be correct.

(b) From given expression, $\omega_d^2 - \omega_0^2 = -\frac{\gamma^2}{4m^2}$

$$\therefore\ -10^{-6}\,\omega_0^2 = -\frac{\gamma^2}{4M^2}$$

$$\Rightarrow\ 4 \times 10^{-6} = \frac{\gamma^2}{\omega_0^2 M^2} = \frac{1}{Q^2}. \qquad \therefore\ Q = \frac{1}{2 \times 10^{-3}} = 500$$

(c) If the body is in equilibrium, $mg = kx \Rightarrow x = \frac{mg}{k} = \frac{g}{\omega^2}$

Note that $\omega^2 = k/m$.

$\therefore$ Potential energy is $\frac{1}{2} m\omega^2 x^2 = \frac{1}{2} m\omega^2 \frac{g^2}{\omega^4} = \frac{mg^2}{2\omega^2}$.

3(a) $$Q = \frac{\omega_0}{\omega_2 - \omega_1} = \frac{\omega_0}{1.02\,\omega_0 - 0.98\,\omega_0} = \frac{1}{0.04} = 25$$

(b) $t^* = \frac{1}{\gamma}$. This is when $E \to E e^{-1}$.

Then, $Q = \frac{2\pi}{\gamma T} = \frac{\omega_0}{\gamma}$. $\therefore\ \gamma = \frac{\omega_0}{Q} = \frac{\omega_0}{25}$.

(c) From the definition of Q factor,

$$\frac{Q}{2\pi} = \frac{\text{Energy stored in the system}}{\text{Energy lost per cycle}}$$

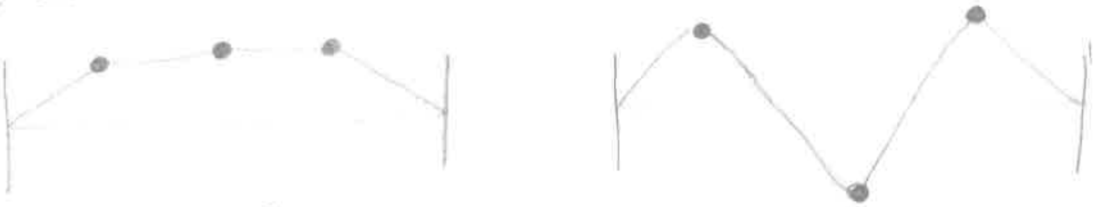
$\therefore$ fraction of energy lost/cycle $= 2\pi/Q = 2\pi/25$

(d) Physically, this graph tell us the range of driving frequencies over which the oscillatory system's response is significantly large.

4(a)

(i) There will be 3 normal modes.

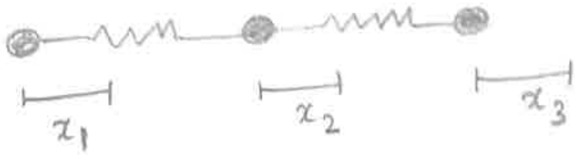
(ii)



Lowest frequency   highest frequency

NOTE: Mirror images of these patterns are also valid.

(b)



Assume $x_2 > x_1$ and $x_3 > x_2$. Each mass m

Then,

$$m\ddot{x}_1 = k(x_2 - x_1),$$
$$m\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2), \qquad \omega_0^2 = \frac{k}{m}$$
$$m\ddot{x}_3 = -k(x_3 - x_2).$$

In matrix form:
$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \begin{pmatrix} -\omega_0^2 & \omega_0^2 & 0 \\ \omega_0^2 & -2\omega_0^2 & \omega_0^2 \\ 0 & \omega_0^2 & -\omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Assume normal mode solutions:
$$x_1 = A e^{i\omega t}, \quad x_2 = B e^{i\omega t}, \quad x_3 = C e^{i\omega t}$$

Substitute these in the matrix equation above.
Rearranging the equation, we get

$$\underbrace{\begin{pmatrix} \omega_0^2 - \omega^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & \omega_0^2 - \omega^2 \end{pmatrix}}_{M} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Non-trivial solution for A, B, C will exist only if
det $M = 0$.

$\det(M) = 0$ gives

$$(\omega_0^2 - \omega^2)\left[(2\omega_0^2 - \omega^2)(\omega_0^2 - \omega^2) - \omega_0^4\right] - \omega_0^4 + \omega_0^2\left[(\omega_0^2 - \omega^2)\,\omega_0^2\right] = 0$$

Solving this for ω^2, gives

$$\omega^2 = 0,\ \omega_0^2,\ 3\,\omega_0^2$$

$\therefore$ Normal mode frequencies: $0,\ \frac{k}{m},\ \frac{3k}{m}$.

5 (a) Assume $x(t) = A \sin \omega t \Rightarrow \sin \omega t = x/A$

$\dot{x}(t) = A\omega \cos \omega t \Rightarrow \cos \omega t = \dot{x}/A\omega$

From this, we get $\frac{x^2}{A^2} + \frac{\dot{x}^2}{A^2\omega^2} = 1$

At any time t_1, (x_1, v_1) is a solution to this equation.

$$\frac{x_1^2}{A^2} + \frac{v_1^2}{A^2\omega^2} = 1 \Rightarrow x_1^2 + \frac{v_1^2}{\omega^2} = A^2$$

Similarly, (x_2, v_2) is also a solution.

$$\therefore \frac{x_2^2}{A^2} + \frac{v_2^2}{A^2\omega^2} = 1 \Rightarrow x_2^2 + \frac{v_2^2}{\omega^2} = A^2$$

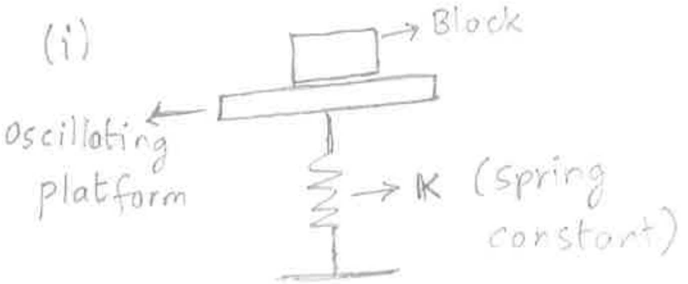
$$\therefore x_1^2 + \frac{v_1^2}{\omega^2} = x_2^2 + \frac{v_2^2}{\omega^2} \Rightarrow \omega^2(x_1^2 - x_2^2) = v_2^2 - v_1^2$$

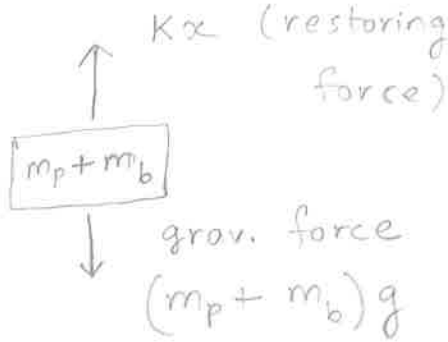
Rearranging this and substituting $\omega = 2\pi/T$, we get

$$T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$$

5(b)

(i)





Let m_p = mass of the platform
m_b = mass of the block

If restoring force exceeds gravitational force, then block will lose contact with the platform; $Kx > (m_p + m_b)g$

Limiting value of x at which contact is lost:

$$x = x_c = \frac{(m_p + m_b)g}{K}$$

Note $\omega^2 = \frac{K}{(m_p + m_b)} \Rightarrow K = \omega^2 (m_p + m_g) \Rightarrow x_c = g/\omega^2$.

Contact will be lost at $x_c = g/\omega^2 \simeq \frac{10}{(2\pi\nu)^2} = 0.025\ m.$

$(g \simeq 10$ and $\pi^2 \simeq 10)$

(ii) $x(t) = A \sin \omega t$, where $A = 0.05 m$.

Starting from $t = 0$, block will lose contact with platform at $t = t^*$

velocity of platform at $t = t^*$: $\dot{x}(t^*) = A\omega \cos(\omega t^*)$

Energy of block at $t = t^*$: $\frac{1}{2} m_b A^2 \omega^2 \cos^2 \omega t^*$

This energy is converted to potential energy.

$\therefore \quad m_b g h = \frac{1}{2} m_b A^2 \omega^2 \cos^2\left[\omega\left(\frac{1}{\omega} \sin^{-1} \frac{x_c}{A}\right)\right] \quad \left(h = 0 \text{ at } x = x_c\right)$

This is because $x_c = A \sin \omega t^*$.

Then, we get $h = \frac{A^2\omega^2}{2g} \cos^2\left(\sin^{-1}\left(\frac{x_c}{A}\right)\right)$

Note, $\frac{x_c}{A} = \frac{1}{2}$. $\therefore \sin^{-1}\left(\frac{x_c}{A}\right) = \frac{\pi}{6}$. $\quad \therefore h = \frac{A^2\omega^2}{2g} \cos^2 \frac{\pi}{6} = 0.0375\ m$

$\therefore$ The reqd. height is $0.025 + 0.0375 - 0.05 = 0.0125\ m$