(1) Harmonic Oscillator using WKB

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 x^2 = E$$

$$x^2 = \frac{2E}{\omega^2} \Rightarrow x_m = \pm\sqrt{\frac{2E}{\omega^2}}$$

$$\oint p\, dx = 2\pi\hbar\left(n+\tfrac{1}{2}\right)$$

$$\int_{-x_m}^{x_m} \sqrt{2E-\omega^2x^2}\, dx = 2\pi\hbar\left(n+\tfrac{1}{2}\right)$$

$$\sqrt{2E}\int\sqrt{1-\frac{\omega^2}{2E}x^2}\, dx$$

$$x = y\sqrt{\frac{2E}{\omega^2}}$$

$$\sqrt{2E}\sqrt{\frac{2E}{\omega^2}}\int_{-1}^{1}\sqrt{1-y^2}\, dy$$

$$\frac{2E}{\omega}\int_{-1}^{1}\sqrt{1-y^2}\, dy$$

$$\frac{2E}{\omega}\cdot\frac{\pi}{2} = 2\pi\hbar\left(n+\tfrac{1}{2}\right)$$

$$E_n = 2\left(n+\tfrac{1}{2}\right)\hbar\omega$$

$$\boxed{E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega}$$

(2) Infinite well

$$H = \frac{p^2}{2m} = E \qquad \left(-\frac{L}{2} \le x \le \frac{L}{2}\right)$$

$$p = \sqrt{2mE}$$

$$\oint_{-L/2}^{L/2}\sqrt{2mE}\, dx = 2\sqrt{2mE}\, L$$

$$\therefore \cancel{2}\sqrt{2mE}\, L = \left(n+\tfrac{1}{2}\right)\cancel{2}\pi\hbar$$

$$2mE = \left(n+\tfrac{1}{2}\right)^2\frac{\pi^2\hbar^2}{L^2}$$

$$\text{or} \quad E = \left(n+\tfrac{1}{2}\right)^2\frac{\pi^2\hbar^2}{2mL^2}$$

$$E = \left(n^2+\frac{1}{4}+2n\right)\frac{\pi^2\hbar^2}{2mL^2}$$

$$= \frac{n^2\pi^2\hbar^2}{2mL^2} + \frac{\pi^2\hbar^2}{2mL^2} + \frac{2n\pi^2\hbar^2}{2mL^2}$$

$$E_n = E_{infwell} + \text{''} \quad \text{''}$$

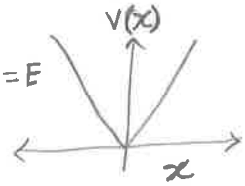
$$= E_{infwell}\left(1 + \frac{(2n+1)\pi^2\hbar^2}{2mL^2 E_{infwell}}\right)$$

$$E_n = E_{infwell}\left(1 + \frac{(2n+1)\pi^2\hbar^2}{2mL^2}\cdot\frac{2mL^2}{n^2\pi^2\hbar^2}\right)$$

$$E_n = E_{infwell}\left(1 + \frac{(2n+1)}{n^2}\right)$$

As $n \to \infty$, $\quad E_n \to E_{infwell}$

③ $H = \frac{p^2}{2m} + k|x| = E$



turning points $x_c = \pm E/k$

$$2\int_{-E/k}^{E/k} \sqrt{2m(E-k|x|)}\,dx = \left(n+\frac{1}{2}\right)2\pi\hbar$$

$$\sqrt{2mk}\oint \sqrt{\frac{E}{k}-|x|}\,dx$$

$$\sqrt{2mk}\int_0^{E/k} 2\left(\frac{E}{k}-x\right)^{1/2} dx = \left(n+\frac{1}{2}\right)\pi\hbar$$

Let $x = \frac{E}{k}y$, $dx = \frac{E}{k}dy$

$$2\sqrt{2mk}\int_0^1 \sqrt{\frac{E}{k}}\sqrt{1-y}\,\frac{E}{k}\,dy = \left(n+\frac{1}{2}\right)\pi\hbar$$

$$2\sqrt{2mk}\,\frac{E^{3/2}}{k\sqrt{k}}\underbrace{\int_0^1\sqrt{1-y}\,dy}_{2/3} = \left(n+\frac{1}{2}\right)\pi\hbar$$

$$\frac{4}{3}\frac{\sqrt{2m}}{k}E^{3/2} = \left(n+\frac{1}{2}\right)\pi\hbar$$

$$E^{3/2} = \frac{3k\pi\hbar}{4\sqrt{2m}}\left(n+\frac{1}{2}\right)$$

$$E = \left(\frac{3k}{4}\frac{\pi\hbar}{\sqrt{2m}}\right)^{2/3}\left(n+\frac{1}{2}\right)^{2/3}$$

$$E = c\left(n+\frac{1}{2}\right)^{2/3}$$

④

$$H = \frac{p^2}{2m} + a|x|^{\alpha} = E \quad (\alpha > 0)$$

$$x_c = \left(\frac{E}{a}\right)^{1/\alpha}$$

$$\cancel{2}\int_{-x_c}^{x_c}\sqrt{2m(E-a|x|^{\alpha})}\,dx = \left(n+\frac{1}{2}\right)\cancel{2}\pi\hbar$$

$$\sqrt{2ma}\int_0^{x_c}\sqrt{\frac{E}{a}-x^{\alpha}}\,dx = \left(n+\frac{1}{2}\right)\pi\hbar$$

$$x = y\left(\frac{E}{a}\right)^{1/\alpha}$$

$$\sqrt{2ma}\int_0^1\sqrt{\frac{E}{a}}\sqrt{1-y^{\alpha}}\,dy\left(\frac{E}{a}\right)^{1/\alpha}$$

$$\sqrt{2ma}\left(\frac{E}{a}\right)^{\frac{1}{2}+\frac{1}{\alpha}}\underbrace{\int_0^1\sqrt{1-y^{\alpha}}\,dy}_{I} = \left(n+\frac{1}{2}\right)\pi\hbar$$

$$E^{\frac{1}{2}+\frac{1}{\alpha}} = \left(n+\frac{1}{2}\right)\frac{\pi\hbar\, a^{\frac{1}{2}+\frac{1}{\alpha}}}{I\sqrt{2ma}}$$

$$E_n = \left(n+\frac{1}{2}\right)^{\frac{2\alpha}{2+\alpha}}\left(\frac{\pi\hbar\, a^{(2+\alpha)/2\alpha}}{I\sqrt{2ma}}\right)^{\frac{2\alpha}{2+\alpha}}$$

$$= a\left(n+\frac{1}{2}\right)^{\frac{2\alpha}{2+\alpha}}\left(\frac{\pi\hbar}{I\sqrt{2ma}}\right)^{2\alpha/2+\alpha}$$

$$E_n = a\left(n+\frac{1}{2}\right)^{\frac{2\alpha}{2+\alpha}}\left(\frac{\hbar}{\sqrt{2ma}}\right)^{2\alpha/2+\alpha}\left(\frac{1}{I}\right)^{\frac{2\alpha}{2+\alpha}}$$

$$I = \sqrt{\pi}\,\frac{\Gamma\left(\frac{1}{\alpha}+1\right)}{\Gamma\left(\frac{1}{\alpha}+\frac{3}{2}\right)}$$

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,dt \quad (\mathrm{Re}(z) > 0)$$

$$\Gamma(n) = (n-1)! \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad \Gamma(1) = 1$$