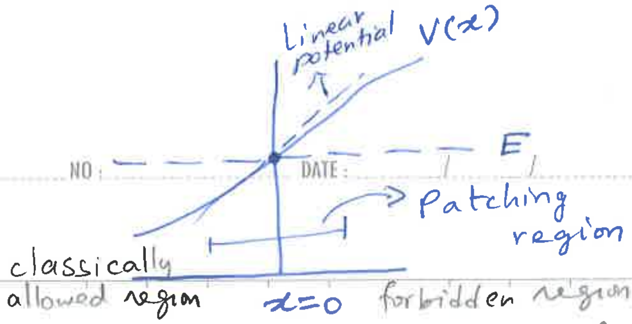
Connection Formula for WKB:



$$\psi(x) = \frac{C}{\sqrt{p(x)}} \, e^{\pm i/\hbar \int^{x} p(x')\, dx'}$$

Let this be right hand turning point

①

$$\psi(x) = \frac{1}{\sqrt{p(x)}}\left[B\, e^{i/\hbar \int_x^0 p(x')\,dx'} + C\, e^{-i/\hbar \int_x^0 p(x')\,dx'}\right] \quad (x<0)$$

$$= \frac{1}{\sqrt{|p(x)|}}\, D\, e^{-\frac{1}{\hbar}\int_0^x |p(x')|\,dx'} \quad (x>0)$$

WKB Solution

② Approximate the potential $V(x)$ by straight line

$$V(x) = V(0) + V'(0)\,x = E + V'(0)\,x$$

For linearized $V(x)$:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_p}{dx^2} + \left(E - V'(0)\,x\right)\psi_p = E\,\psi_p$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_p}{dx^2} - V'(0)\,x\,\psi_p = 0 \implies \frac{d^2\psi_p}{dx^2} = \frac{2m\,V'(0)}{\hbar^2}\,x\,\psi_p$$

put $\alpha^3 = \frac{2m\,V'(0)}{\hbar^2}$ (or) $\alpha = \left(\frac{2m\,V'(0)}{\hbar^2}\right)^{1/3}$

change variable to $z = \alpha x$

Then, $\frac{d^2\psi_p}{dz^2} = z\,\psi_p$ → Airy's equation

Solutions are Airy function

$$\psi_p(x) = a\,Ai(\alpha x) + b\,Bi(\alpha x)$$

This is valid in the vicinity of $x=0$.

③ Overlap region: where WKB joins the Airy function

$$p(x) = \sqrt{2m(E - V(x))} = \sqrt{2m(E - E - V'(0)\,x)} = \sqrt{2m(-V'(0)\,x)}$$

$$p(x) = \sqrt{\frac{2m\,V'(0)}{\hbar^2}}\cdot\hbar = \hbar\,\alpha^{3/2}\sqrt{-x}$$

(4) Let's consider overlap region-2 (forbidden region)

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WKB soln is $\psi(x) = \dfrac{D}{\sqrt{|p(x)|}} e^{-\frac{1}{\hbar}\int_0^x |p(x')|\, dx'}$ $(x>0)$

$p(x) = \hbar \alpha^{3/2} \sqrt{-x}$, $|p(x)| = \hbar \alpha^{3/2} \sqrt{x}$

$$\int_0^x |p(x')|\, dx' = \int_0^x \hbar \alpha^{3/2} \sqrt{x'}\, dx' = \frac{2}{3}\hbar (\alpha x)^{3/2}$$

$\therefore$ WKB soln is $\psi(x) = \dfrac{D}{\sqrt{\hbar \alpha^{3/2}\sqrt{x}}} e^{-\frac{1}{\hbar}\frac{2}{3}\hbar(\alpha x)^{3/2}} = \dfrac{D}{\sqrt{\hbar}\alpha^{3/4} x^{1/4}} e^{-\frac{2}{3}(\alpha x)^{3/2}}$ —— ①

(5) Patching function ψ_p in region-2:

Use asymptotic form for Airy fn: for $(z>>0)$ $Ai(z) \sim \dfrac{1}{2\sqrt{\pi} z^{1/4}} e^{-\frac{2}{3}z^{3/2}}$, $Bi(z) \sim \dfrac{1}{\sqrt{\pi} z^{1/4}} e^{\frac{2}{3}z^{3/2}}$ for $(z>>0)$

$$\therefore \psi_p \simeq \frac{a}{2\sqrt{\pi}(\alpha x)^{1/4}} e^{-\frac{2}{3}(\alpha x)^{3/2}} + \frac{b}{\sqrt{\pi}(\alpha x)^{1/4}} e^{\frac{2}{3}(\alpha x)^{3/2}} \quad —— ②$$

Comparing ① & ②:

$$a = \sqrt{\frac{4\pi}{\alpha\hbar}}\, D \quad \text{and} \quad b = 0$$

(6) Repeat steps ④ & ⑤ for region-1 (allowed region)

WKB soln:

$$\psi(x) = \frac{1}{\sqrt{\hbar}\alpha^{3/4}(-x)^{1/4}}\left[B e^{i\frac{2}{3}(-\alpha x)^{3/2}} + C e^{-i\frac{2}{3}(-\alpha x)^{3/2}}\right] \quad —— ③$$

Patching function ψ_p in region-1: Use asymptotic form for $z << 0$

$$\psi_p(x) \simeq \frac{a}{\sqrt{\pi}(-\alpha x)^{1/4}} \sin\left[\frac{2}{3}(-\alpha x)^{3/2} + \frac{\pi}{4}\right]$$

Note that $b=0$

$$\psi_p(x) = \frac{a}{\sqrt{\pi}(-\alpha x)^{1/4}} \frac{1}{2i}\left[e^{i\pi/4} e^{i\frac{2}{3}(-\alpha x)^{3/2}} - e^{-i\pi/4} e^{-i\frac{2}{3}(-\alpha x)^{3/2}} \right] \quad (4)$$

Comparing Eq (3) and (4):

$$B = -i e^{i\pi/4} D \quad \text{and} \quad C = i e^{-i\pi/4} D$$

(7) Substituting for B and C in WKB solution, we get

$$\psi(x) = \frac{2D}{\sqrt{p(x)}} \sin\left(\frac{1}{\hbar}\int_x^{x_2} p(x')\,dx' + \frac{\pi}{4}\right) \quad \text{if } x < x_2$$

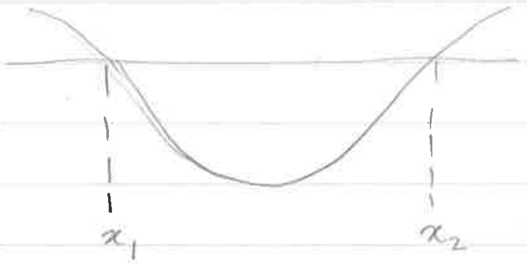
x_2: Turning point

$$= \frac{D}{\sqrt{|p(x)|}} \exp\left[-\frac{1}{\hbar}\int_{x_2}^{x} |p(x')|\,dx'\right] \quad \text{if } x > x_2$$

These are the connecting formulas.

(8) Example

We want solution for $x_1 \le x \le x_2$



$\overset{x}{\longleftrightarrow}$ Turning points

We can write WKB connection formula as

$$\psi(x) = \frac{2D}{\sqrt{p(x)}} \sin \theta_2(x)$$

where $\theta_2(x) = \frac{1}{\hbar}\int_x^{x_2} p(x')\,dx' + \pi/4$

Another possibility is

$$\psi(x) = -\frac{2D}{\sqrt{p(x)}} \sin \theta_1(x) \quad \text{where} \quad \theta_1(x) = \frac{1}{\hbar}\int_{x_1}^{x} p(x')\,dx' - \frac{\pi}{4}$$

$$\theta_2 - \theta_1 = n\pi \Rightarrow \frac{1}{\hbar}\int_x^{x_2} p(x')\,dx' + \frac{\pi}{4} + \frac{1}{\hbar}\int_{x_1}^{x} p(x')\,dx' + \pi/4 = n\pi$$

$$\frac{1}{\hbar}\int_{x_1}^{x_2} p(x')\,dx' = n\pi - \frac{\pi}{2} \Rightarrow \oint p(x)\,dx = 2\pi\hbar\left(n - \tfrac{1}{2}\right) \quad n = 1, 2, 3 \ldots$$

$$= 2\pi\hbar\left(n + \tfrac{1}{2}\right) \quad n = 0, 1, 2 \ldots$$

About Airy function

SUBJECT: $\frac{d^2y}{dz^2} = zy$ NO: DATE: / /

Solution:

$$Ai(z) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{s^3}{3} + sz\right) ds$$

$$Bi(z) = \frac{1}{\pi}\int_0^{\infty} e^{-s^3/3 + sz} + \sin\left(\frac{s^3}{3} + sz\right) ds$$

Asymptotic forms:

$$Ai(z) \sim \frac{1}{2\sqrt{\pi}\, z^{1/4}} e^{-\frac{2}{3} z^{3/2}}$$

$$Bi(z) \sim \frac{1}{\sqrt{\pi}\, z^{1/4}} e^{\frac{2}{3} z^{3/2}}$$

(for $z >> 0$)