Sample Questions: Quantum Mechanics II

Q1) Given that $|+\rangle = |s = 1/2, m = 1/2\rangle = \binom{1}{0}$ and $|-\rangle = \left|s = \frac{1}{2}, m = -\frac{1}{2}\right\rangle = \binom{0}{1}$. Find explicit forms for the direct product states $|++\rangle, |+-\rangle$.

Soln:

Single-particle spin states:

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we want to construct product states for a two-particle spin system.

Compute $|++\rangle$

$$|++\rangle = |+\rangle \otimes |+\rangle$$

Using the given matrix representations:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The tensor product is computed as:

$$|++\rangle = \begin{pmatrix} 1 \cdot 1 \\ 1 \cdot 0 \\ 0 \cdot 1 \\ 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus, the explicit form of $|++\rangle$ is:

$$|++\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Compute $|+-\rangle$

The state $|+-\rangle$ represents the first particle in the $|+\rangle$ state and the second particle in the $|-\rangle$ state:

$$|+-\rangle = |+\rangle \otimes |-\rangle$$

Using the matrix representations:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The tensor product is computed as:

$$|+-\rangle = \begin{pmatrix} 1 \cdot 0 \\ 1 \cdot 1 \\ 0 \cdot 0 \\ 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Thus, the explicit form of $|+-\rangle$ is:

$$|+-\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Similarly, the remaining states $|-+\rangle$ and $|--\rangle$ are:

$$|-+\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |--\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

These states form the standard basis for a two-spin system in the S_z basis.

Q2) Clebsch-Gordan Coefficients for $\frac{1}{2} \otimes 1 = \frac{3}{2} \oplus \frac{1}{2}$

Soln:

Uncoupled basis states:

$$\left|\frac{1}{2}, m_1\right\rangle \otimes |1, m_2\rangle, \tag{1}$$

where $m_1 = \pm\frac{1}{2}$ and $m_2 = -1, 0, 1$. These states form a six-dimensional space, which can be rewritten in terms of total angular momentum eigenstates $|J, M\rangle$, using CG coefficients:

$$|J, M\rangle = \sum_{m_1, m_2} C^{J,M}_{m_1,m_2} \left|\frac{1}{2}, m_1\right\rangle \otimes |1, m_2\rangle. \tag{2}$$

Possible J and M Values

$$J = \frac{3}{2}, \quad J = \frac{1}{2}. \tag{3}$$

For each J, the allowed M values are:

- When $J = \frac{3}{2}$:

$$M = \frac{3}{2}, \quad \frac{1}{2}, \quad -\frac{1}{2}, \quad -\frac{3}{2}.$$

- When $J = \frac{1}{2}$:

$$M = \frac{1}{2}, \quad -\frac{1}{2}.$$

These coupled states can be expressed as linear combinations of the uncoupled basis states using Clebsch-Gordan coefficients:

$$|J, M\rangle = \sum_{m_1, m_2} C^{J,M}_{m_1,m_2} \left|\frac{1}{2}, m_1\right\rangle \otimes |1, m_2\rangle.$$

1) $|J, M\rangle = \left|\frac{3}{2}, \frac{3}{2}\right\rangle$

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes |1, 1\rangle.$$

2) $|J, M\rangle = \left|\frac{3}{2}, \frac{1}{2}\right\rangle$

The total angular momentum lowering operator S_- satisfies the relation:

$$S_-|J, M\rangle = \sqrt{(J+M)(J-M+1)}|J, M-1\rangle.$$

Applying S_- to $|3/2, 3/2\rangle$

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes |1, 1\rangle .$$

Applying the lowering operator:

$$S_- \left|\frac{3}{2}, \frac{3}{2}\right\rangle = \sqrt{3} \left|\frac{3}{2}, \frac{1}{2}\right\rangle .$$

total spin operator S_- is the sum of the lowering operators acting on each particle,

$$S_- = S_{1-} + S_{2-}.$$

apply this to the product state:

$$(S_{1-} + S_{2-}) \left(\left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes |1, 1\rangle \right) .$$

Applying S_- to Each Component Using the action of the lowering operator:

$$S_{1-} \left|\frac{1}{2}, \frac{1}{2}\right\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle , \tag{4}$$

$$S_{2-} |1, 1\rangle = \sqrt{2} |1, 0\rangle . \tag{5}$$

Thus, applying S_-:

$$S_- \left|\frac{3}{2}, \frac{3}{2}\right\rangle = S_{1-} \left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes |1, 1\rangle + \left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes S_{2-} |1, 1\rangle$$

$$= \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \otimes |1, 1\rangle + \sqrt{2} \left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes |1, 0\rangle .$$

Comparing with

$$S_- \left|\frac{3}{2}, \frac{3}{2}\right\rangle = \sqrt{3} \left|\frac{3}{2}, \frac{1}{2}\right\rangle ,$$

we equate terms and obtain the Clebsch-Gordan coefficients:

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}} \left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes |1, 0\rangle + \sqrt{\frac{1}{3}} \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \otimes |1, 1\rangle . \tag{6}$$

3) $|J, M\rangle = \left|\frac{3}{2}, -\frac{1}{2}\right\rangle$
Lowering from $M = \frac{1}{2}$ to $M = -\frac{1}{2}$
Applying the lowering operator S_- to $\left|\frac{3}{2}, \frac{1}{2}\right\rangle$:

$$S_- \left|\frac{3}{2}, \frac{1}{2}\right\rangle = \sqrt{2} \left|\frac{3}{2}, -\frac{1}{2}\right\rangle . \tag{7}$$

Since $S_- = S_{1-} + S_{2-}$, we act on:

$$(S_{1-} + S_{2-}) \left(\sqrt{\frac{2}{3}} \left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes |1, 0\rangle + \sqrt{\frac{1}{3}} \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \otimes |1, 1\rangle \right) . \tag{8}$$

Using the known actions:

$$S_{1-}\left|\frac{1}{2},\frac{1}{2}\right\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle, \tag{9}$$

$$S_{2-}\left|1,0\right\rangle = \sqrt{2}\left|1,-1\right\rangle, \tag{10}$$

$$S_{2-}\left|1,1\right\rangle = \sqrt{2}\left|1,0\right\rangle. \tag{11}$$

Applying these:

$$S_{-}\left|\frac{3}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|1,0\right\rangle\right) + \sqrt{\frac{2}{3}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \sqrt{2}\left|1,-1\right\rangle\right) \tag{12}$$

$$+\sqrt{\frac{1}{3}}\left(\sqrt{2}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|1,0\right\rangle\right). \tag{13}$$

Simplifying:

$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|1,0\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|1,-1\right\rangle. \tag{14}$$

4) $|J,M\rangle = \left|\frac{3}{2},-\frac{3}{2}\right\rangle$

Lowering from $M=-\frac{1}{2}$ to $M=-\frac{3}{2}$
Applying S_- to $|3/2,-1/2\rangle$:

$$S_{-}\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle. \tag{15}$$

Since $S_- = S_{1-} + S_{2-}$, we act on:

$$(S_{1-}+S_{2-})\left(\sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|1,0\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|1,-1\right\rangle\right). \tag{16}$$

Using the known actions:

$$S_{1-}\left|\frac{1}{2},-\frac{1}{2}\right\rangle = 0, \tag{17}$$

$$S_{2-}\left|1,0\right\rangle = \sqrt{2}\left|1,-1\right\rangle, \tag{18}$$

$$S_{2-}\left|1,-1\right\rangle = 0. \tag{19}$$

Thus:

$$S_{-}\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \sqrt{2}\left|1,-1\right\rangle. \tag{20}$$

Simplifying:

$$\left|\frac{3}{2},-\frac{3}{2}\right\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|1,-1\right\rangle. \tag{21}$$

- For $J=\frac{1}{2}$ States

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes |1,0\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,1\rangle$$
$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,0\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes |1,-1\rangle$$

Clebsch-Gordan Coefficients:

J, M	(m_1, m_2)	CG Coefficient
$\frac{3}{2},\frac{3}{2}$	$(\frac{1}{2},1)$	1
$\frac{3}{2},\frac{1}{2}$	$(\frac{1}{2},0)$	$\sqrt{\frac{2}{3}}$
	$(-\frac{1}{2},1)$	$\sqrt{\frac{1}{3}}$
$\frac{3}{2},-\frac{1}{2}$	$(-\frac{1}{2},0)$	$\sqrt{\frac{2}{3}}$
	$(\frac{1}{2},-1)$	$\sqrt{\frac{1}{3}}$
$\frac{3}{2},-\frac{3}{2}$	$(-\frac{1}{2},-1)$	1
$\frac{1}{2},\frac{1}{2}$	$(\frac{1}{2},0)$	$\sqrt{\frac{1}{3}}$
	$(-\frac{1}{2},1)$	$-\sqrt{\frac{2}{3}}$
$\frac{1}{2},-\frac{1}{2}$	$(-\frac{1}{2},0)$	$\sqrt{\frac{1}{3}}$
	$(\frac{1}{2},-1)$	$-\sqrt{\frac{2}{3}}$

Q3) If $H = -\left(\gamma_1\vec{S}_1 + \gamma_2\vec{S}_2\right)\cdot\vec{B}$ and $\vec{B} = B_0\hat{k}$, then find the eigenvalues of H using direct product basis.

Soln: Since $\vec{B}$ is in the z-direction, only the S_z components contribute:

$$H = -B_0\left(\gamma_1 S_{1z} + \gamma_2 S_{2z}\right).$$

The spin states of each particle are given by:

$$S_{1z}|s_1, m_1\rangle = m_1|s_1, m_1\rangle, \quad S_{2z}|s_2, m_2\rangle = m_2|s_2, m_2\rangle.$$

Thus in product basis $|m_1, m_2\rangle$ the Hamiltonian acts as:

$$H|m_1, m_2\rangle = -B_0(\gamma_1 m_1 + \gamma_2 m_2)|m_1, m_2\rangle.$$

Product Basis: For two spin-$\frac{1}{2}$ particles, the possible values of m_1 and m_2 are:

$$m_1, m_2 \in \left\{\frac{1}{2}, -\frac{1}{2}\right\}.$$

$$|++\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle, \quad |+-\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle, \quad |-+\rangle = \left|-\frac{1}{2},\frac{1}{2}\right\rangle, \quad |--\rangle = \left|-\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Applying the Hamiltonian to each basis state

$$H|++\rangle = -B_0\left(\gamma_1\cdot\frac{1}{2} + \gamma_2\cdot\frac{1}{2}\right)|++\rangle$$
$$= -\frac{B_0}{2}(\gamma_1+\gamma_2)|++\rangle.$$

$$E_{++} = -\frac{B_0}{2}(\gamma_1+\gamma_2), \quad \Longrightarrow \text{Eigenvalue of } |++\rangle.$$

Similarly

$$H|+-\rangle = -B_0\left(\gamma_1\cdot\frac{1}{2}+\gamma_2\cdot\left(-\frac{1}{2}\right)\right)|+-\rangle$$

$$= -\frac{B_0}{2}(\gamma_1-\gamma_2)|+-\rangle.$$

$$E_{+-} = -\frac{B_0}{2}(\gamma_1-\gamma_2) \quad \Longrightarrow \text{Eigenvalue of } |+-\rangle.$$

For $|-+\rangle$:

$$H|-+\rangle = -B_0\left(\gamma_1\cdot\left(-\frac{1}{2}\right)+\gamma_2\cdot\frac{1}{2}\right)|-+\rangle$$

$$= -\frac{B_0}{2}(-\gamma_1+\gamma_2)|-+\rangle.$$

Eigenvalue for $|-+\rangle$ is:

$$E_{-+} = -\frac{B_0}{2}(-\gamma_1+\gamma_2) = \frac{B_0}{2}(\gamma_1-\gamma_2).$$

For $|--\rangle$:

$$H|--\rangle = -B_0\left(\gamma_1\cdot\left(-\frac{1}{2}\right)+\gamma_2\cdot\left(-\frac{1}{2}\right)\right)|--\rangle$$

$$= -\frac{B_0}{2}(-\gamma_1-\gamma_2)|--\rangle.$$

Eigenvalue for $|--\rangle$ is:

$$E_{--} = \frac{B_0}{2}(\gamma_1+\gamma_2).$$

Basis State	Eigenvalue
$\vert ++\rangle$	$-\frac{B_0}{2}(\gamma_1+\gamma_2)$
$\vert +-\rangle$	$-\frac{B_0}{2}(\gamma_1-\gamma_2)$
$\vert -+\rangle$	$\frac{B_0}{2}(\gamma_1-\gamma_2)$
$\vert --\rangle$	$\frac{B_0}{2}(\gamma_1+\gamma_2)$

Q4) The Hamiltonian describing the hyperfine interactions is $H_{hf} = A\vec{S}_1\cdot\vec{S}_2$ $(A>0)$. The total Hamiltonian $H = H_{\text{coloumb}} + H_{hf}$ Show that H_{hf} splits the ground state into two levels

$$E_+ = -R_y + \frac{\hbar^2 A}{4} \text{ and } E_- = -R_y - \frac{3\hbar^2 A}{4}$$

soln:

The Hamiltonian describing the hyperfine interaction is:

$$H_{hf} = A\vec{S}_1\cdot\vec{S}_2, \quad A>0.$$

The total Hamiltonian is given by:

$$H = H_{\text{Coulomb}} + H_{hf}.$$

Expressing H_{hf} in Terms of Total Spin $\vec{S}$

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(S^2 - S_1^2 - S_2^2\right).$$

Since S_1 and S_2 are spin-$\frac{1}{2}$ particles:

$$S_1^2 = S_2^2 = \frac{3}{4}\hbar^2.$$

Thus,

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(S^2 - \frac{3}{2}\hbar^2\right).$$

Energy Eigenvalues for Singlet and Triplet States

The total spin S can take values:

- Triplet states ($S = 1$): $S^2 = 1(1+1)\hbar^2 = 2\hbar^2$.
- Singlet state ($S = 0$): $S^2 = 0$.

For the triplet states:

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(2\hbar^2 - \frac{3}{2}\hbar^2\right) = \frac{1}{4}\hbar^2.$$

For the singlet state:

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(0 - \frac{3}{2}\hbar^2\right) = -\frac{3}{4}\hbar^2.$$

Since $H_{hf} = A\vec{S}_1 \cdot \vec{S}_2$, the energy corrections due to H_{hf} are:

$$E_{\text{triplet}} = A \times \frac{1}{4}\hbar^2 = \frac{\hbar^2 A}{4}.$$

$$E_{\text{singlet}} = A \times \left(-\frac{3}{4}\hbar^2\right) = -\frac{3\hbar^2 A}{4}.$$

Thus, the total energy levels including the Coulomb contribution ($E_0 = -R_y$) are:

$$E_+ = -R_y + \frac{\hbar^2 A}{4}, \quad E_- = -R_y - \frac{3\hbar^2 A}{4}.$$

Q5) Let H of a two spin system be given by,

$$H = A + B \quad \vec{S}_1 \cdot \vec{S}_2 + C\left(S_{1z} + S_{2z}\right), \quad A, B, C \text{ are constants.}$$

Determine the eigenvalues and eigenfunction if the particles have spin 1/2.

Soln:

Expressing $\vec{S}_1 \cdot \vec{S}_2$ in Terms of Total Spin

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(S^2 - S_1^2 - S_2^2\right).$$

Since each spin has $S_1^2 = S_2^2 = \frac{3}{4}\hbar^2$, this simplifies to:

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(S^2 - \frac{3}{2}\hbar^2\right).$$

Basis States:

The two spin-1/2 particles form a singlet and a triplet:

- **Triplet States** ($S = 1$, symmetric):

$$|1,1\rangle = |\uparrow\uparrow\rangle,$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right),$$

$$|1,-1\rangle = |\downarrow\downarrow\rangle.$$

- **Singlet State** ($S = 0$, antisymmetric):

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right).$$

Evaluating $\vec{S}_1 \cdot \vec{S}_2$

Using $S^2|S, m_S\rangle = S(S+1)\hbar^2|S, m_S\rangle$:

- For the triplet states ($S = 1$):

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(1(1+1) - \frac{3}{2}\right)\hbar^2 = \frac{1}{4}\hbar^2.$$

- For the singlet state ($S = 0$):

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(0(0+1) - \frac{3}{2}\right)\hbar^2 = -\frac{3}{4}\hbar^2.$$

Evaluating $S_{1z} + S_{2z}$:

$$(S_{1z} + S_{2z})|S, m_S\rangle = m_S\hbar|S, m_S\rangle,$$

we obtain the following results for each state:

$$(S_{1z} + S_{2z})|1,1\rangle = \hbar|1,1\rangle,$$

$$(S_{1z} + S_{2z})|1,0\rangle = 0,$$

$$(S_{1z} + S_{2z})|1,-1\rangle = -\hbar|1,-1\rangle,$$

$$(S_{1z} + S_{2z})|0,0\rangle = 0.$$

The Hamiltonian acts on the basis states as:

$$H|S, m_S\rangle = \left(A + B\vec{S}_1 \cdot \vec{S}_2 + Cm_S\hbar\right)|S, m_S\rangle.$$

Using our previously calculated values for $\vec{S}_1 \cdot \vec{S}_2$, we find the eigenvalues:
For the triplet states ($S = 1$)

- For $|1,1\rangle$:

$$E_{1,1} = A + \frac{B}{4}\hbar^2 + C\hbar.$$

- For $|1,0\rangle$:

$$E_{1,0} = A + \frac{B}{4}\hbar^2.$$

- For $|1,-1\rangle$:

$$E_{1,-1} = A + \frac{B}{4}\hbar^2 - C\hbar.$$

For the singlet state ($S=0$): $|1,0\rangle$

$$E_{0,0} = A - \frac{3B}{4}\hbar^2.$$

Q6) Consider $H^1 = \lambda x^4$ and $H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$. Show that the first order energy shift is $E_n^1 = \frac{3\hbar^2\lambda}{4m^2w^2}\left(1+2n+2n^2\right)$.

Soln: First Order Energy Shift for $H^1 = \lambda x^4$. We consider the unperturbed Hamiltonian:

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

The perturbation is given by:

$$H^1 = \lambda x^4.$$

<u>First Order Energy Correction Formula</u>

In non-degenerate perturbation theory, the first-order energy correction is given by:

$$E_n^1 = \langle n|H^1|n\rangle.$$

Thus, we need to compute:

$$E_n^1 = \lambda\langle n|x^4|n\rangle.$$

Expressing x in Terms of Ladder Operators. The position operator x in terms of the creation ($a^\dagger$) and annihilation (a) operators is given by:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger).$$

Squaring both sides:

$$x^2 = \frac{\hbar}{2m\omega}(a+a^\dagger)^2.$$

Expanding:

$$x^2 = \frac{\hbar}{2m\omega}\left(a^2 + a^{\dagger 2} + a^\dagger a + aa^\dagger\right).$$

Using the commutator relation:

$$[a,a^\dagger] = 1 \quad \Rightarrow \quad aa^\dagger = a^\dagger a + 1,$$

$$x^2 = \frac{\hbar}{2m\omega}\left(a^2 + a^{\dagger 2} + 2a^\dagger a + 1\right).$$

$$x^4 = \left(\frac{\hbar}{2m\omega}\right)^2 (a^2 + a^{\dagger 2} + 2a^\dagger a + 1)^2.$$

Expanding:

$$x^4 = \left(\frac{\hbar}{2m\omega}\right)^2 \left(a^4 + a^{\dagger 4} + 4a^\dagger a + 4a^\dagger a a^\dagger a + 2(a^2 a^\dagger a + a^\dagger a a^2) + 2(a^{\dagger 2} a^\dagger a + a^\dagger a a^{\dagger 2}) + 2(a^2 + a^{\dagger 2}) + 1\right).$$

Taking expectation value in state $|n\rangle$, we use:

$$\langle n|a^\dagger a|n\rangle = n,$$
$$\langle n|a^2|n\rangle = 0, \quad \langle n|a^{\dagger 2}|n\rangle = 0,$$
$$\langle n|a^\dagger a a^\dagger a|n\rangle = n(n+1).$$

Since $a^2|n\rangle$ and $a^{\dagger 2}|n\rangle$ are orthogonal to $|n\rangle$, their expectation values vanish.

Thus, only the terms involving n survive:

$$\langle n|x^4|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^2 (1 + 2n + 2n^2)3.$$

Substituting this into our formula for E_n^1:

$$E_n^1 = \lambda \frac{3\hbar^2}{4m^2\omega^2}(1 + 2n + 2n^2).$$
