Jan 2025 semester - test 1 - Brief solutions

1a) Allowed bands: particles or usually electrons in this energy band can travel freely through the periodic lattice.
Forbidden bands: particles with energy in this band can not travel through the periodic lattice.

1b) In the limit that energy is large (*E>>1),* then width of forbidden band goes to zero. This is seen in the figure as well.

1c) Largest allowed energy in first allowed band

$$\sqrt{\frac{2mE}{\hbar}} = \pi \Rightarrow \frac{2mE}{\hbar^2} = \pi^2 \Rightarrow E = \frac{\pi^2\hbar^2}{2mE}$$

2) $s_1 = \begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$ $\qquad s_2 = \begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$

$s_1 \otimes s_2$ gives 4 possibilities:

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

3) $S_z = S_{1z} + S_{2z}$,

$$|{+}{+}\rangle = |+\rangle_1 \otimes |+\rangle_2 = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\ \tfrac{1}{2}\rangle_2$$
$$= |\tfrac{1}{2}\ \tfrac{1}{2};\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

We need $(S_{1z} + S_{2z})\,|\tfrac{1}{2}\ \tfrac{1}{2};\tfrac{1}{2}\ \tfrac{1}{2}\rangle$

$$= \left(S_{1z}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_1\right)\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_2 + \left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_1 S_{2z}\left(\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_2\right)$$

$$= \frac{\hbar}{2}\left|\tfrac{1}{2}\,\tfrac{1}{2},\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \frac{\hbar}{2}\left|\tfrac{1}{2}\,\tfrac{1}{2},\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle$$

$$= \hbar\left|\tfrac{1}{2}\,\tfrac{1}{2},\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle = \hbar\,|++\rangle$$

4) $$H = -\vec{\mu}\cdot\vec{B} = \frac{ge}{2m_e}\vec{S}\cdot\vec{B} = \frac{e\hbar}{2m_e}\vec{\sigma}\cdot\vec{B}$$

($g=1$ for $\vec{\mu}$ from spin).

Given that $\vec{B} = B_0\hat{k}$

$$\therefore\ H = \frac{e\hbar}{2me}\sigma_z B_0 = \frac{e\hbar B_0}{2me}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

H is already in diagonal form.

$\therefore$ Using the notation $\mu_B = \dfrac{e\hbar}{2me}$

Energy eigenvalues are: $\mu_B B_0$ and $-\mu_B B_0$.

5) $$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \quad |\beta\rangle = e^{-\frac{1}{2}|\beta|^2}\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!}}|n\rangle$$

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)}\sum_{n=0}^{\infty}\frac{\alpha^n\beta^{*n}}{n!} = e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)}e^{\alpha^*\beta}$$

$$|\langle\alpha|\beta\rangle|^2 = e^{-\left(|\alpha|^2+|\beta|^2\right)+\left(\alpha\beta^*+\alpha^*\beta\right)}$$

$$= \exp\left(-|\alpha-\beta|^2\right).$$