PH 3214 (Jan, 2025 semester) Endsem exam solutions

1) $s_1 = 1 \quad , \quad s_2 = 1/2$

product states

$|s_1 m_1 ; s_2 m_2\rangle$

$|1, -1 ; \frac{1}{2}, -\frac{1}{2}\rangle$

$|1, -1 ; \frac{1}{2}, \frac{1}{2}\rangle$

$|1, 0 ; \frac{1}{2}, -\frac{1}{2}\rangle$

$|1, 0 ; \frac{1}{2}, \frac{1}{2}\rangle$

$|1, 1 ; \frac{1}{2}, -\frac{1}{2}\rangle$

$|1, 1 ; \frac{1}{2}, \frac{1}{2}\rangle$

coupled states

$|s\ m ; s_1\ s_2\rangle = |s, m\rangle$

$|3/2, -3/2\rangle$

$|3/2, -1/2\rangle$

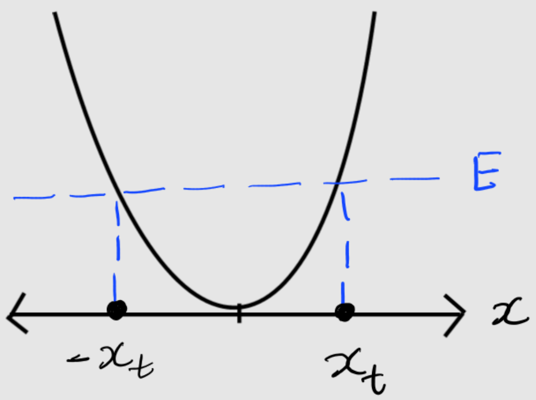
$|3/2, 1/2\rangle$

$|3/2, 3/2\rangle$

$|1/2, -1/2\rangle$

$|1/2, 1/2\rangle$

2)



$E = \alpha x^4$

$x_t \rightarrow$ Turning points

$x_t = \pm \left(\frac{E}{\alpha}\right)^{1/4}$

3) First order energy correction $E_n^1 = V_0 \langle n | x^2 | n\rangle$

$|n\rangle \rightarrow$ normalised eigenstate of any ID system

Energy of state $|n\rangle$: $\mathcal{E}_n = E_n + E_n^1$

$$\mathcal{E}_n = E_n + V_0 \langle n | x^2 | n \rangle$$

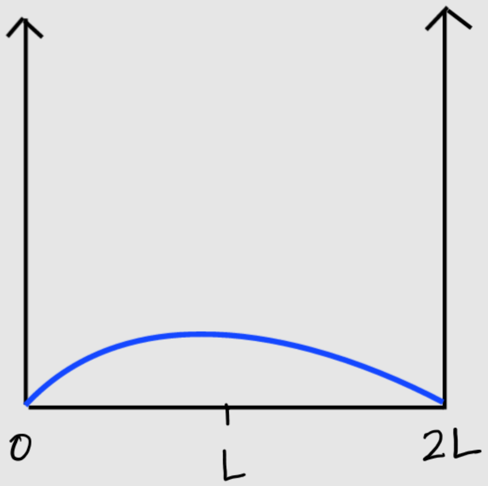
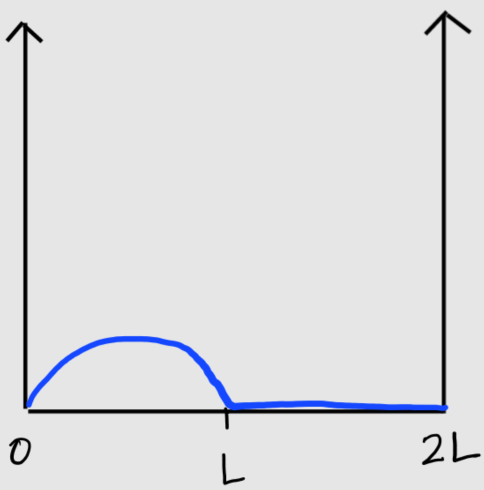
This result is valid if $|V_0 x^2|$ is much smaller than H_0 (unperturbed system).

4) Three identical fermions represented by states

$$\psi_{n_1}(x), \quad \psi_{n_2}(x) \quad \text{and} \quad \psi_{n_3}(x)$$

$$\Psi(x_1, x_2, x_3) = N \begin{vmatrix} \psi_{n_1}(x_1) & \psi_{n_2}(x_1) & \psi_{n_3}(x_1) \\ \psi_{n_1}(x_2) & \psi_{n_2}(x_2) & \psi_{n_3}(x_2) \\ \psi_{n_1}(x_3) & \psi_{n_2}(x_3) & \psi_{n_3}(x_3) \end{vmatrix}$$

N = normalisation constant $= \frac{1}{\sqrt{3!}} = \frac{1}{\sqrt{6}}$.

5) (a)  (b) 

6) Let $H|n\rangle = E_n|n\rangle$.

ψ is some trial unnormalised state

$$\langle\psi|H|\psi\rangle = \sum_{n=0}^{\infty} \langle\psi|H|n\rangle\langle n|\psi\rangle = \sum_{n=0} E_n \langle\psi|n\rangle\langle n|\psi\rangle$$

Note that $\sum_n |\langle\psi|n\rangle|^2 = \langle\psi|\psi\rangle$ ---- ①

$$\therefore \langle\psi|H|\psi\rangle = E_0 |\langle\psi|0\rangle|^2 + \sum_{n>0} E_n |\langle\psi|n\rangle|^2$$

From Eq. (1):

$$|\langle\psi|0\rangle|^2 + \sum_{n>0} |\langle\psi|n\rangle|^2 = \langle\psi|\psi\rangle$$

$$\langle\psi|H|\psi\rangle = E_0\left[\langle\psi|\psi\rangle - \sum_{n>0} |\langle\psi|n\rangle|^2\right] + \sum_{n>0} E_n |\langle\psi|n\rangle|^2$$

$$= E_0 \langle\psi|\psi\rangle + \underbrace{\sum_{n>0} (E_n - E_0) |\langle\psi|n\rangle|^2}_{\text{This is positive definite}}$$

$$\therefore \langle\psi|H|\psi\rangle \geq E_0 \langle\psi|\psi\rangle$$

$$\Rightarrow \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0$$

7) We need WKB formula

$$\oint p(x)\, dx = \left(n + \frac{1}{2}\right)\hbar\omega$$

For $H = \frac{p^2}{2m} + \alpha|x| = E$

Turning points are $\pm x_t = \pm(E/\alpha)$

$$\therefore \quad 2\int_{-x_t}^{x_t} \sqrt{2m(E - \alpha|x|)}\, dx = \left(n + \frac{1}{2}\right) 2\pi\hbar$$

Factor of 2 is required to account for one full periodic orbit between $-x_t$ to x_t.

$$\sqrt{2m\alpha}\; 2\int_0^{E/\alpha} \left(\frac{E}{\alpha} - x\right)^{1/2} dx = \left(n + \frac{1}{2}\right)\pi\hbar$$

put $x = \frac{E}{\alpha} y$, $\Rightarrow$ $dx = \frac{E}{\alpha} dy$

$$\therefore \quad 2\sqrt{2m\alpha} \int_0^1 \sqrt{\frac{E}{\alpha}}\, \sqrt{1-y}\; \frac{E}{\alpha}\, dy = \left(n + \frac{1}{2}\right)\pi\hbar$$

$$2\sqrt{2m\alpha}\; \frac{E^{3/2}}{\alpha\sqrt{\alpha}} \underbrace{\int_0^1 \sqrt{1-y}\, dy}_{2/3} = \left(n + \frac{1}{2}\right)\pi\hbar$$

$$\therefore \quad E_n = \left(\frac{3\alpha}{4} \frac{\pi\hbar}{\sqrt{2m}}\right)^{2/3} \left(n + \frac{1}{2}\right)^{2/3}$$

8.

(a) symmetric state

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}} \left[\varphi_1(x_1) \ \varphi_2(x_2) + \varphi_1(x_2) \ \varphi_2(x_1) \right]$$

$$= \frac{1}{\sqrt{2}} \frac{2}{L} \left[\sin \frac{\pi x_1}{L} \ \sin \frac{2\pi x_2}{L} + \sin \frac{\pi x_2}{L} \ \sin \frac{2\pi x_1}{L} \right]$$

(b) if $x_1 = x_2 = x$, then

$$\psi(x) = \frac{4}{\sqrt{2}L} \left(\sin \frac{\pi x}{L} \ \sin \frac{2\pi x}{L} \right)$$

$$P_s(x) = |\psi(x)|^2$$ → probability density

$$= \frac{8}{L^2} \sin^2\left(\frac{\pi x}{L}\right) \sin^2\left(\frac{2\pi x}{L}\right)$$

Let's put $\theta = \pi x / L$, and $c = 8/L^2$. Then,

$$P_s = c \ \sin^2\theta \ \sin^2 2\theta$$

(c) $P_s(x)$ evaluated at $x = L/3$ gives

$$P_s(L/3) = \frac{8}{L^2} \sin^2\left(\frac{\pi}{3}\right) \sin^2\left(\frac{2\pi}{3}\right) = \frac{8}{L^2} \left(\frac{\sqrt{3}}{2}\right)^2 \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{9}{4L^2}$$

NOTE : There was an error in part (c) of this question. Marks were given if the procedure was right.

9) $V(x) = \frac{1}{2} m\omega^2 x^2$ and $H^1 = A x^3 e^{-\gamma t}$

Transition probability under time-dependent perturbation

$$d_f(t) = -\frac{i}{\hbar} \int_0^t \langle f^0 | H | i^0 \rangle e^{i\omega_{fi} t'} dt'$$

$|i^0\rangle = |0\rangle$ is the initial state

$|f^0\rangle = |n\rangle$ is the final state

$$\omega_{fi} = \frac{E_n - E_0}{\hbar} = \frac{1}{\hbar}\left(n + \frac{1}{2} - \frac{1}{2}\right)\hbar\omega$$

$$= n\omega$$

$$d_f(t) = -\frac{i}{\hbar} A \langle n | x^3 | 0 \rangle \int_0^t e^{-\gamma t'} e^{in\omega t'} dt'$$

Performing the integral, we get

$$d_f(t) = -\frac{i}{\hbar} A \langle n | x^3 | 0 \rangle \left[\frac{e^{(in\omega - \gamma)t} - 1}{in\omega - \gamma} \right]$$

In the limit $t \to \infty$, $e^{-\gamma t} \to 0$.

$$\therefore d_f(t) = \frac{-iA}{\hbar} \langle n | x^3 | 0 \rangle \left[\frac{-1}{(in\omega - \gamma)} \right]$$

$$P_{0\to n} = |d_f(t)|^2$$

$$= \frac{A^2}{\hbar^2} |\langle n | x^3 | 0 \rangle|^2 \frac{1}{(\gamma^2 + n^2\omega^2)}$$

$$\langle n | x^3 | 0 \rangle = \left(\frac{\hbar}{2m\omega} \right)^{3/2} \left[\sqrt{6}\ \delta_{n,3} + 3\ \delta_{n,1} \right]$$

$$P_{0\to n} = \frac{A^2}{\hbar^2} \frac{\hbar^3}{8m^3\omega^3} \left[\sqrt{6}\ \delta_{n,3} + 3\ \delta_{n,1} \right]^2 \frac{1}{(\gamma^2 + n^2\omega^2)}$$

Note that $\left(\sqrt{6}\ \delta_{n,3} + 3\ \delta_{n,1} \right)^2 = 6\ \delta_{n,3} + 9\ \delta_{n,1}$

since $\delta_{n,3}\ \delta_{n,1} = 0.$

$$P_{0\to n} = \frac{A^2\hbar}{8m^3\omega^3} \left[6\ \delta_{n,3} + 9\ \delta_{n,1} \right] \frac{1}{\gamma^2 + n^2\omega^2}$$

$$P_{0\to n} = \frac{3A^2\hbar}{8m^3\omega^3} \cdot \frac{\left[2\ \delta_{n,3} + 3\ \delta_{n,1} \right]}{(\gamma^2 + n^2\omega^2)}$$