1) A collection of independent electrons have spins polarised parallel to uniform magnec field of magnitude B_0 pointing in z direction. A perturbing field in x-direction is applied. What is the Condition under which adiabatic there would apply.

Perturbing field in x-direction $\Rightarrow B'(t) = 10^{-3} B_0 \left(1 - e^{-t/\tau}\right) \qquad t \geq 0$

$$H = -\vec{\mu} \cdot \vec{B}$$

$$= -\vec{\mu} \cdot \left(B_0 \hat{k} + B_0 10^{-3} \hat{i} \left(1 - e^{-t/\tau}\right)\right) \qquad t \geq 0$$

$$H = \underbrace{-\vec{\mu} \cdot \hat{k} B_0}_{H^0} \underbrace{- B_0 10^{-3} \left(1 - e^{-t/\tau}\right) \vec{\mu} \cdot \hat{i}}_{H^1(t)}$$

Larmor freq: $\Omega = \dfrac{|e| B_0}{m_e}$

$$T = \frac{2\pi}{\Omega} = \frac{2\pi m_e}{|e| B_0}$$

if $\tau \gg \dfrac{2\pi m_e}{|e| B_0}$, then adiabatic theorem would apply.

2) Hydrogen atom in ground state is placed in uniform electric field in z-direction.

$$\mathcal{E} = \mathcal{E}_0 e^{-t/\tau} \leftarrow \text{electric field}$$

This is turned on at $t=0$. What is the probability that atom is excited to 2p state at $t >> \tau$.

$$H = \frac{p^2}{2m} - \frac{e^2}{r} + e\mathcal{E}z$$

$$H = \underbrace{\frac{p^2}{2m} - \frac{e^2}{r}}_{H^0} + \underbrace{e\mathcal{E}_0 z e^{-t/\tau}}_{H^1(t)}$$

$$d_f(t) = -\frac{i}{\hbar} \int \langle f^0 | H^1 | i^0 \rangle e^{-i\omega_{fi}t'} dt' \; e^{-t/\tau}$$

$$= -\frac{ie\mathcal{E}_0}{\hbar} \int_0^t \langle f^0 | z | i^0 \rangle e^{-\left(\frac{1}{\tau} + i\omega_{fi}\right)t'} dt'$$

$$|i^0\rangle = \psi_{100} = \sqrt{\frac{1}{\pi a_0^3}} e^{-r/a_0} \quad \Big\} \text{ 1s state}$$

$$\left.\begin{aligned} |f^0\rangle = \underline{\underline{\psi_{210}}} &= \sqrt{\frac{1}{32\pi a_0^3}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta \\ = \psi_{21\pm1} &= \mp \sqrt{\frac{1}{64\pi a_0^3}} \frac{r}{a_0} e^{-r/2a_0} \sin\theta \, e^{\pm i\varphi} \end{aligned}\right\} \text{ 2p state}$$

$$\Rightarrow \langle \psi_{100} | z | \psi_{210} \rangle = \int \sqrt{\frac{1}{\pi a_0^3}} \sqrt{\frac{1}{32\pi a_0^3}} \frac{r}{a_0} e^{-\frac{3}{2}\frac{r}{a_0}} z \cos\theta \underbrace{r^2 dr \sin\theta \, d\theta \, d\varphi}_{\substack{\text{Jacobian for} \\ \text{transforming from} \\ (x,y,z) \to (r,\theta,\varphi)}}$$

$$z = r\cos\theta$$

$$= \frac{1}{\sqrt{32}} \frac{1}{\pi a_0^3} \int \frac{r}{a_0} e^{-\frac{3}{2}\frac{r}{a_0}} r\cos\theta \cos\theta \; r^2 dr \sin\theta \, d\theta \, d\varphi$$

$$= \frac{2\pi}{\sqrt{32}\,\pi a_0^3} \int_0^\infty \frac{r^4}{a_0} e^{-\frac{3}{2}\frac{r}{a_0}} dr \int_0^\pi \cos^2\theta \sin\theta \, d\theta$$

$$= \frac{a_0^4}{\sqrt{8}} \qquad \frac{4}{9} \qquad \frac{2}{3}$$

$$\langle \psi_{100} | z | \psi_{210} \rangle = \frac{a_0^4}{\sqrt{8}} \frac{4}{9} \times \frac{2}{3} = \frac{a_0^4}{\sqrt{2}} \frac{4}{27}$$

$$d_f(t) = \frac{-ie\mathcal{E}_0}{\hbar} \frac{a_0^4}{\sqrt{2}} \frac{4}{27} \int_0^t e^{-\left(\frac{1}{\tau} + i\omega_{fi}\right)t'} dt'$$

$$d_f(t) = \frac{-ie\mathcal{E}_0}{\hbar} \frac{a_0^4}{\sqrt{2}} \frac{4}{27} \left. \frac{e^{-\left(\frac{1}{\tau} + i\omega_{fi}\right)t'}}{\left(\frac{1}{\tau} + i\omega_{fi}\right)} \right|_0^t \quad \text{----} (1)$$

$$P_{100 \to 210} = |d_f(t)|^2$$

Exercise: Compute $P_{100 \to 210}$ using Eq (1).

However, $P_{100 \to 2,1,\pm 1} = 0$ since $\int_0^{2\pi} e^{\pm i\varphi} \, d\varphi = 0$

3) A 1D harmonic oscillator has spring constant suddenly reduced by a factor of 1/2. The oscillator is initially in ground state. Show that the oscillator remains in ground state with high probability?

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 = \frac{p^2}{2m} + \frac{k}{2} x^2 \qquad m\omega^2 = k$$

After sudden change $k \to \frac{k}{2}$

$$H' = \frac{p^2}{2m} + \frac{k}{4} x^2$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} = \left(\frac{\sqrt{mk}}{\pi\hbar}\right)^{1/4} e^{-\frac{\sqrt{mk}}{2\hbar}x^2}$$

$$\psi_0'(x) = \left(\frac{\sqrt{mk/2}}{\pi\hbar}\right)^{1/4} e^{-\frac{\sqrt{mk/2}}{2\hbar}x^2}$$

$$\left|\langle \psi_0(x) \,|\, \psi_0'(x) \rangle\right|^2 = \int_{-\infty}^{\infty} \left(\frac{\sqrt{mk}}{\pi\hbar}\right)^{1/4} \left(\frac{\sqrt{mk/2}}{\pi\hbar}\right)^{1/4} \underbrace{e^{-\frac{\sqrt{mk}}{2\hbar}x^2}\, e^{-\frac{\sqrt{mk/2}}{2\hbar}x^2}}_{\text{This is a Gaussian integral \& doable.}}\, dx$$

Probability of finding perturbed system in ground state

$$= \frac{2^{5/4}}{1+\sqrt{2}} \simeq 0.985$$