× Periodic perturbation

example: atom interacting with monochromatic light beam

atom placed between plates of a capacitor connected to ac source

Let $H'(t) = H' e^{-i\omega t}$

Our system begins to be perturbed by H' at time $t=0$.

Transition amplitude from $|i^\circ\rangle$ to $|f^\circ\rangle$ in time t is

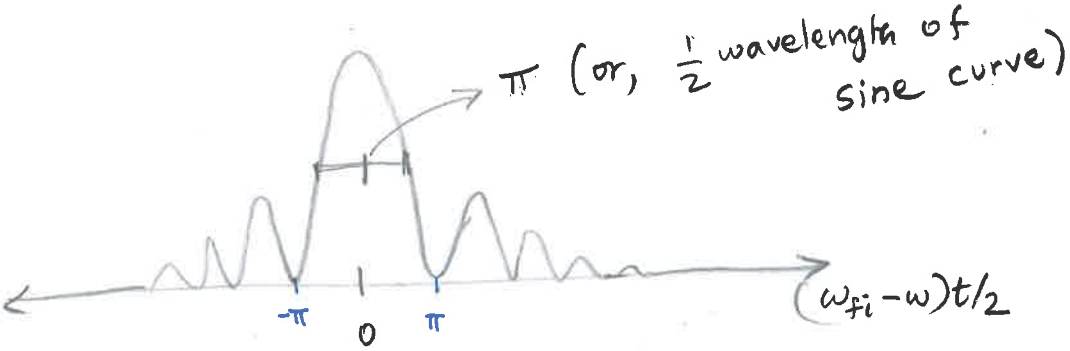
$$d_f(t) = \left(\frac{-i}{\hbar}\right)\int_0^t \langle f^\circ|H'|i^\circ\rangle\, e^{-i\omega t'}\, e^{i\omega_{fi}t'}\, dt'$$

$$= -\frac{i}{\hbar}\langle f^\circ|H'|i^\circ\rangle \left.\frac{e^{i(\omega_{fi}-\omega)t'}}{i(\omega_{fi}-\omega)}\right|_0^t$$

$$= \frac{-1}{\hbar}\langle f^\circ|H'|i^\circ\rangle \frac{\left[e^{i(\omega_{fi}-\omega)t}-1\right]}{(\omega_{fi}-\omega)}$$

$$d_f(t) = -\frac{1}{\hbar}\langle f^\circ|H'|i^\circ\rangle\, e^{i(\omega_{fi}-\omega)t/2}\frac{\left[e^{i(\omega_{fi}-\omega)t/2}-e^{-i(\omega_{fi}-\omega)t/2}\right]}{(\omega_{fi}-\omega)t/2}\,\frac{t}{2}$$

$$|d_f(t)|^2 = \frac{1}{\hbar^2}|\langle f^\circ|H'|i^\circ\rangle|^2 \left(\frac{\sin(\omega_{fi}-\omega)t/2}{(\omega_{fi}-\omega)t/2}\right)^2 t^2 \quad \text{-----} \; (1)$$



$$\therefore \left|(\omega_{fi}-\omega)\frac{t}{2}\right| \lesssim \pi$$

System will go to states $|f\rangle$ under this condition

$$\left(\frac{E_f^0 - E_i^0}{\hbar}\right)\frac{t}{2} = \pm\pi + \frac{\omega t}{2}$$

$$E_f^0 t - E_i^0 t = \pm 2\pi\hbar + \hbar\omega t$$

$$\therefore \quad E_f^0 t = (E_i^0 t + \hbar\omega t) \pm 2\pi\hbar$$

or, $$E_f^0 - E_i^0 = \hbar\omega\left(1 \pm \frac{2\pi}{\omega t}\right)$$

- For small t, E_f^0 is not the preferred final state.
- For $\omega t \gg 2\pi$, $E_f^0 = E_i^0 + \hbar\omega$ is the most favoured. We need few cycles of perturbation for this to happen.

What happens as $t \to \infty$?

Consider system exposed to perturbation from $t = -\frac{T}{2}$ to $\frac{T}{2}$ and $T \to \infty$. Then,

$$d_f = \lim_{T\to\infty} \frac{-i}{\hbar} \int_{-T/2}^{T/2} H'_{fi} \, e^{i(\omega_{fi} - \omega)t'} \, dt'$$

$$= -\frac{i}{\hbar} H'_{fi} \, 2\pi \, \delta(\omega_{fi} - \omega)$$

$$P_{i\to f} = |d_f|^2 = \frac{4\pi^2}{\hbar^2} |H'_{fi}|^2 \, \delta(\omega_{fi} - \omega) \, \delta(\omega_{fi} - \omega)$$

$$\delta(\omega_{fi} - \omega)\,\delta(\omega_{fi} - \omega) = \lim_{T\to\infty} \delta(\omega_{fi} - \omega) \frac{1}{2\pi} \int_{-T/2}^{T/2} e^{i(\omega_{fi} - \omega)t} \, dt$$

$$= \lim_{T\to\infty} \delta(\omega_{fi} - \omega) \frac{T}{2\pi}$$

$$\therefore \quad P_{i\to f} = \frac{4\pi^2}{\hbar^2} |H'_{fi}|^2 \, \frac{T}{2\pi} \, \delta(E_f^0 - E_i^0 - \hbar\omega) \, \hbar$$

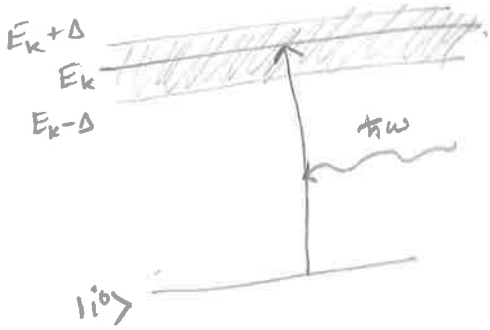
Transition rate $\quad R_{i\to f} = \frac{P_{i\to f}}{T} = \frac{2\pi}{\hbar} \left| \langle f^0 | H' | i^0 \rangle \right|^2 \delta(E_f^0 - E_i^0 - \hbar\omega)$

This is Fermi's Golden rule. Has many applications.

- Due to δ-function in the formula, transition rate makes sense only under an integral.
- As $t \to \infty$, Fourier transform of the perturbation is sharply at the frequency of perturbation (due to δ-function). System "sees" a single frequency.

In short-time limit: use Eq (1), and expand sine term

$$P_{i\to f} = \frac{t^2 \, |H'_{fi}|^2}{\hbar^2}.$$ Transition rate: $R_{i\to f} = \frac{t}{\hbar^2} |H'_{fi}|^2$



<u>Excited states lie in a band</u>

$dN = g(E_k)\, dE_k$
↳ density of final states

The prob. of transition occurs to a state in a band of width 2Δ centred at E_k

$$\bar{P}_{i\to f} = \int_{E_f-\Delta}^{E_f+\Delta} P_{i\to f} \, g(E_f)\, dE_f$$

Substitute for $P_{i\to f}$ from Eq (1):

$$\bar{P}_{i\to f} = \int_{E_f-\Delta}^{E_f+\Delta} dE_f \; g(E_f) \frac{|\langle f^0 | H' | i^0 \rangle|^2}{\hbar^2} \frac{\sin^2\beta}{\beta^2/t^2}$$

where $2\hbar\beta = \hbar(\omega_{fi} - \omega)t = (E_f - E_i - \hbar\omega)t$

For fixed $|i^0\rangle$, t and ω, we have $dE_f = \frac{2\hbar}{t} d\beta$

$$\bar{P}_{i\to f} = \frac{2t}{\hbar} \int_{-\delta}^{\delta} g(E_f) \, |H'_{fi}|^2 \frac{\sin^2\beta}{\beta^2} d\beta,$$

δ is the value of β corresponding to Δ for E_f.

$$\bar{P}_{i\to f} = \frac{2t}{\hbar}\, g(E_f)\, |H'_{fi}|^2 \int_{-\infty}^{\infty} \frac{\sin^2\beta}{\beta^2}\, d\beta$$

assumption: $g(E_f)$ and $|H'_{fi}|^2$ vary slowly with β.

$$= \left[\frac{2\pi}{\hbar}\, g(E_f)\, |H'_{fi}|^2\right] t$$

The associated transition probability rate

$$\bar{R}_{i\to f} = \frac{2\pi}{\hbar}\, g(E_f)\, |H'_{fi}|^2$$