Problem: Find CG coefficients for $\frac{1}{2} \otimes 1 = \frac{3}{2} \oplus \frac{1}{2}$

$s_1 \otimes s_2 \qquad s \oplus s$

$s_1 \otimes s_1 = s \oplus s$

$2s_1+1=2$	$2s+1=4$
$2s_2+1=3$	$2s+1=2$
$3\times 2=6$	$4+2=6$

dim. of Hilbert Space : 6

direct product states

$|s_1\, m_1\ s_2\, m_2\rangle$

$\underbrace{|\tfrac{1}{2}\ \pm\tfrac{1}{2}\rangle}_{s_1}, \quad \underbrace{|1, \pm 1\rangle, |1, 0\rangle}_{s_2}$

Coupled states

$|s\, m;\, s_1\, s_2\rangle$

$s = s_1 - s_2$ to $s_1 + s_2$

$= 1 - 1/2$ to $1 + 1/2$

$= \frac{1}{2}$ to $\frac{3}{2}$

$s = \frac{1}{2}, \frac{3}{2}$

m

$-\frac{1}{2}, \frac{1}{2} \qquad -\frac{3}{2}\ \ -\frac{1}{2}\ \ \frac{1}{2}\ \ \frac{3}{2}$

$$\left.\begin{array}{ll} |\tfrac{1}{2}, \tfrac{1}{2}; 1, -1\rangle & |\tfrac{1}{2}, -\tfrac{1}{2}; 1, -1\rangle \\ |\tfrac{1}{2}, \tfrac{1}{2}; 1, 0\rangle & |\tfrac{1}{2}, -\tfrac{1}{2}; 1, 0\rangle \\ |\tfrac{1}{2}, \tfrac{1}{2}; 1, 1\rangle & |\tfrac{1}{2}, -\tfrac{1}{2}; 1, 1\rangle \end{array}\right\} \text{Product states}$$

$$\text{Coupled states} \left\{\begin{array}{ll} |\tfrac{3}{2}, \tfrac{3}{2}\rangle & \\ |\tfrac{3}{2}, \tfrac{1}{2}\rangle & |\tfrac{1}{2}, \tfrac{1}{2}\rangle \\ |\tfrac{3}{2}, -\tfrac{1}{2}\rangle & |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \\ |\tfrac{3}{2}, -\tfrac{3}{2}\rangle & \end{array}\right.$$

$$|\tfrac{3}{2}, \tfrac{3}{2}\rangle = |\tfrac{1}{2}, \tfrac{1}{2}; 1\, 1\rangle$$

$$|\tfrac{3}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2}, \tfrac{1}{2}; 1, 0\rangle + \sqrt{\tfrac{1}{3}}\,|\tfrac{1}{2}, -\tfrac{1}{2}; 1, 1\rangle$$

$$|\tfrac{3}{2}, -\tfrac{1}{2}\rangle = \overset{\sqrt{1/3}}{\alpha}\,|\tfrac{1}{2}, \tfrac{1}{2}; 1, -1\rangle + \overset{\sqrt{2/3}}{\beta}\,|\tfrac{1}{2}, -\tfrac{1}{2}; 1, 0\rangle$$

$$|\tfrac{3}{2}, -\tfrac{3}{2}\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}; 1, -1\rangle$$

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle = -\sqrt{\tfrac{1}{3}}\,|\tfrac{1}{2}, \tfrac{1}{2}; 1, 0\rangle + \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2}, -\tfrac{1}{2}; 1, 1\rangle$$

$$|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \alpha_1\,|\tfrac{1}{2}, \tfrac{1}{2}; 1, -1\rangle + \beta_1\,|\tfrac{1}{2}, -\tfrac{1}{2}; 1, 0\rangle$$

Lower half to be determined using the relation

$$\langle j_1 m_1, j_2 m_2 | j m\rangle = (-1)^{j_1+j_2-j} \langle j_1 -m_1, j_2 -m_2 | j -m\rangle$$

For example: For $|\frac{3}{2}, -\frac{1}{2}\rangle$

$j_1 = \frac{1}{2}$, $m_1 = \frac{1}{2}$, $j_2 = 1$, $m_2 = -1$

$j = \frac{3}{2}$, $m = -\frac{1}{2}$

$$\langle \tfrac{1}{2}\ \tfrac{1}{2}, 1\ -1 | \tfrac{3}{2}\ -\tfrac{1}{2}\rangle = (-1)^{1+\frac{1}{2}-\frac{3}{2}} \langle \tfrac{1}{2}\ -\tfrac{1}{2}, 1\ 1 | \tfrac{3}{2}\ \tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{1}{3}}$$

Now, assemble all the relation between product states and coupled states in matrix-vector form. Elements of matrix are CG coefficients.

Addition rules for angular momentum l

Let's say N electrons with ang. mom. values $l_1\ l_2\ l_3 \ldots l_N$ — $l_1\ l_2$

$l_1 \le l_2$

ordered $l_1 \le l_2 \le \ldots \le l_N$

Let $l_{sum} = \sum_{i=1}^{N-1} l_i$

Rule is: $l_{min} = \max(0, l_N - l_{sum})$ — $|l_2 - l_1|$

$l_{max} = \sum_{i=1}^{N} l_i$ — $l_1 + l_2$

Possible Total ang mom (squared) values

$$L^2 = \hbar^2 l(l+1)$$

$l = |l_{max}|, |l_{max} - 1|, \ldots l_{min}.$

How many states: $\sum_{l_{min}}^{l_{max}} 2l+1$. You can calculate possible values of L.

Example: 2 p orbital electrons, one f orbital electron

Symbol	s	p	d	f	g	h	...
l	0	1	2	3	4	5	...

Spectrospic notation

$l_1 = 1, \quad l_2 = 1, \quad l_3 = 3$ $\qquad l_{sum} = 1 + 1 = 2$

$l_{min} = \max(0, 3-2) = 1$

$l_{max} = 5$

$\therefore \; l = 1, 2, 3, 4, 5$

Dim. of Hilbert Space: $3 + 5 + 7 + 9 + 11 = 35$

Curse of dimensionality.

- Thankfully $Y_l^m(\theta, \varphi)$ are still the solution.

$l = 1 \qquad m = -1, 0, 1$

In position representation, $Y_l^m(\theta, \varphi)$ are the solution

ex: $\langle \theta, \varphi | l \, m, l_1 \, l_2 \rangle = Y_l^m(\theta, \varphi) \rightarrow$ spherical harmonics.