Addition of angular momentum

- General problem: $\vec{J} = J_1 + J_2$ $\quad \vec{J}_1 = \vec{L}_1 + \vec{S}_1$, $\quad \vec{J}_2 = \vec{L}_2 + \vec{S}_2$
  - What are the eigenvalues & eigenfns of J?

Let's follow the same recipe as for $\vec{S} = \vec{S}_1 + \vec{S}_2$.

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

Bases $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$.

i.e $|s_1\, m_1; s_2\, m_2\rangle$

This is called product basis.

$S_z |s_1\, m_1; s_2\, m_2\rangle = m\hbar\, |s_1\, m_1; s_2\, m_2\rangle$

what are possible values of m?

$$m = -1, \underbrace{0, 0}_{\text{degenerate}}, 1$$

Note: $m_1 = \pm\frac{1}{2}$ $\quad m_2 = \pm\frac{1}{2}$

$(-\frac{1}{2}, \frac{1}{2})$ $\quad (-\frac{1}{2}, \frac{1}{2})$

- $m = m_1 + m_2 = -1, 0, 0, 1$

obtained as $m_1 + m_2$.

- $\overset{\to -2}{-s_1 - s_2}$ and $s_1 + s_2 = 2$ are non-degenerate.
- others are degenerate

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

Product basis: $|j_1\, m_1; j_2\, m_2\rangle = |j_1\, m_1\rangle \otimes |j_2\, m_2\rangle$

$J_z |j_1\, m_1; j_2\, m_2\rangle = m\hbar\, |j_1\, m_1; j_2\, m_2\rangle$

What are the possible values of m?

$$m = j_1 + j_2$$

Consider an example:

Let $j_1 = 1$, $\quad j_2 = 1$

What's the order of J_z matrix?

ans: 9

$m_1 = -j_1, -j_1+1, \ldots, 0, \ldots, j_1-1, j_1$

$m_2 = -j_2, -j_2+1, \ldots, 0, \ldots, j_2-1, j_2$

- $m = m_1 + m_2 = \begin{matrix} -2, & -1, & 0 \\ -1, & 0, & 1 \\ 0, & 1, & 2 \end{matrix}$

- -2 & 2, i.e, $-j_1 - j_2$ and $j_1 + j_2$ are non-degenerate

For degenerate eigenspace of S_z (or J_z), we must find a basis that diagonalises S^2 (or J^2).

Construction of S^2 (and J^2) is tedious.

Let us try out an efficient method based on what we did for $\vec{S}$.

- what are the possible values of s?

$$s = 0, 1, 1, 1$$

i.e values lie in the range $s_1 - s_2$ to $s_1 + s_2$

How many product kets: for $s_1 = 1/2$, $s_2 = 1/2$

$$\sum_{s=s_1-s_2}^{s_1+s_2} 2s+1 = \sum_{s=0}^{1} 2s+1 = 1+3 = 4$$

$$\underbrace{s_1 \otimes s_2}_{(2s_1+1)(2s_2+1)} = 1 \oplus 0$$

$$= (s_1+s_2) \oplus (s_1 - s_2)$$

Hence, we got for total-s basis

$|s, m; s_1 s_2\rangle$ with

$$s_1 + s_2 \geq s \geq s_1 - s_2$$

and $\quad s \geq m \geq -s$

what are the possible values of j?

will take values (assuming $j_1 \geq j_2$)

$$j_1 + j_2,\ j_1 + j_2 - 1, ----, j_1 - j_2$$

# of product kets:

$$\sum_{j=j_1-j_2}^{j_1+j_2} 2j+1 = \sum_{j=0}^{j_1+j_2} 2j+1 - \sum_{j=0}^{j_1-j_2+1} 2j+1$$

$$= (2j_1+1)(2j_2+1)$$

use $\sum_{n=0}^{N} n = \frac{N(N+1)}{2}$

This is similar to $(2s_1+1)(2s_2+1)$.

In some sense, proof of our approach.

So, $j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1)$

$$\oplus ---- \oplus (j_1 - j_2)$$

For $j_1 = 1$, $j_2 = 1$ case:

$$3 \otimes 3 = 2 \oplus 1 \oplus 0$$

$$9 = 5 + 3 + 1$$

Total-j basis are

$|j, m; j_1 j_2\rangle$ with $j_1 + j_2 \geq j \geq j_1 - j_2$

$$j \geq m \geq -j$$

Write the basis states of total-s
$|s, m; s_1\ s_2\rangle$

$s = 0, 1$

$\therefore$ write only $|s, m\rangle$

$$\left.\begin{array}{ll} |1,1\rangle & \\ |1,0\rangle & |0,0\rangle \\ |1,-1\rangle & \end{array}\right\} \quad (1)$$

$s=1$ Subspace $s=0$ Subspace

Write basis states of total-j

$|j, m; j_1\ j_2\rangle$

write only $|j, m\rangle$

- Ex: $j_1 = 1$, $j_2 = 1$ case

Then, $2 \geq j \geq 0$

i.e $j = 0, 1, 2$

$m = -j$ to j.

- Write explicitly the total-j basis states

$$\begin{array}{lll} |2,-2\rangle & & \\ |2,-1\rangle & |1,-1\rangle & \\ |2,0\rangle & |1,0\rangle & |0,0\rangle \\ |2,-1\rangle & |1,-1\rangle & \\ |2,-2\rangle & & \end{array}$$

$j=2$ subspace $j=1$ Subspace $j=0$ Subspace

- Here is the general form of total-j basis states

$j \rightarrow$

$m \downarrow$

$$\begin{array}{llll}
|j_1+j_2, j_1+j_2\rangle & & & \\
|j_1+j_2, j_1+j_2-1\rangle & |j_1+j_2-1, j_1+j_2-1\rangle & --- & \\
|j_1+j_2, j_1+j_2-2\rangle & |j_1+j_2-1, j_1+j_2-2\rangle & --- & |j_1-j_2, j_1-j_2\rangle \\
\vdots & \vdots & & \vdots \\
 & \vdots & --- & \\
 & & & |j_1-j_2, -(j_1-j_2)\rangle \\
|j_1+j_2, -(j_1+j_2-1)\rangle & |j_1+j_2-1, -(j_1+j_2-2)\rangle & ---- & \\
 & |j_1+j_2-1, -(j_1+j_2-1)\rangle & ---- & \\
|j_1+j_2, -(j_1+j_2)\rangle & & & \\
j_1+j_2 \text{ subspace} & j_1+j_2-1 \text{ Subspace} & ---- & j_1-j_2 \text{ subspace}
\end{array}$$

Now, the important question:

Recall, for $\vec{S} = \vec{S}_1 + \vec{S}_2$ problem, we expressed the basis states in Eq (1) in terms of product basis states, i.e, $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$.

- Can we do the same now for $\vec{J} = \vec{J}_1 + \vec{J}_2$ case? How to do it?

we want to write $|j, m, j_1 j_2\rangle$ as linear combinations of $|j_1 m_1 \; j_2 m_2\rangle$ states (or, product states).

Let's revisit our $s_1 = 1/2$, $s_2 = 1/2$ example

Total s- states		Product states
$\|1,1\rangle$	$\longrightarrow$	$\|++\rangle$
$\|1,0\rangle$	$\longrightarrow$	$\frac{1}{\sqrt{2}}(\|+-\rangle + \|-+\rangle)$
$\|1,-1\rangle$	$\longrightarrow$	$\|--\rangle$
$\|0,0\rangle$	$\longrightarrow$	$\frac{1}{\sqrt{2}}(\|+-\rangle - \|-+\rangle)$

For convenience, let's have m for product basis

$\|++\rangle$	$\|+-\rangle$	$\|-+\rangle$	$\|--\rangle$
$m=1$	$m=0$	$m=0$	$m=-1$

(a) for $|1,1\rangle$, $m=1$ and there is only one basis state $|++\rangle$ which has $m=1$. $\therefore |1,1\rangle = |++\rangle$

(b) for $|1,-1\rangle$, $m=-1$ and there is only one basis state $|--\rangle$ which has $m=-1$. $\therefore |1,-1\rangle = |--\rangle$

(c) for $|1,0\rangle$ and $|0,0\rangle$, both have $m=0$. There are two product basis states $|+-\rangle$ and $|-+\rangle$ which have $m=0$.

(a) and (b) are easy. How to find the correct linear combination for (c)?

To do this, let's use

$$J_\pm |j, m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\ |j, m\pm 1\rangle$$

To get $|1,0\rangle$, we have

$$S_-|1,1\rangle = \sqrt{2}\hbar\,|1,0\rangle$$

$$\Rightarrow \quad |1,0\rangle = \frac{1}{\sqrt{2}\hbar} S_-|1,1\rangle = \frac{1}{\sqrt{2}\hbar} S_-|++\rangle .$$

$$|1,0\rangle = \frac{1}{\sqrt{2}\hbar}\left(S_{1-} + S_{2-}\right)|++\rangle = \frac{1}{\sqrt{2}\hbar}\left(\hbar|-+\rangle + \hbar|+-\rangle\right)$$

Details: $S_{1-}|++\rangle = S_{1-}|+\rangle \otimes |+\rangle = S_{1-}|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |+\rangle$

$= \hbar|\tfrac{1}{2},\tfrac{1}{2}-1\rangle \otimes |+\rangle = \hbar|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |+\rangle$

$= \hbar|-\rangle \otimes |+\rangle = \hbar|-+\rangle$

$$\therefore\ |1,0\rangle = \frac{1}{\sqrt{2}}\left(|-+\rangle + |+-\rangle\right)$$

- Next, to find product basis expression for $|1,-1\rangle$, we can lower $|1,0\rangle$, i.e, $S_-|1,0\rangle$. OR, note that there is only one product state with $m=0$, namely, $|--\rangle$.

- To find product basis expression for $|0,0\rangle$:
  This involves both $|+-\rangle$ and $|-+\rangle$.
  We can find the right combination using these conditions
  Since $m=0$ is a subspace, basis vectors of this space should be orthonormal:

$$\therefore\ |1,0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)$$

$$|0,0\rangle = \alpha|+-\rangle + \beta|-+\rangle$$

Our conditions $\langle 0,0|1,0\rangle = 0 \Rightarrow \frac{1}{\sqrt{2}}(\alpha+\beta) = 0$ and $\langle 0,0|0,0\rangle = 1 \Rightarrow \alpha^2+\beta^2 = 1$

Solving these equations, we get $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = -\frac{1}{\sqrt{2}}$

- Now, back to the general problem of $\vec{J} = \vec{J}_1 + \vec{J}_2$

  i.e expressing $|j, m, j_1 j_2\rangle$ in terms of $|j_1 m_1, j_2 m_2\rangle$

- Top state in the first column:

$$\underbrace{|j_1 + j_2}_{j}, \underbrace{j_1 + j_2\rangle}_{m} = |j_1 j_1, j_2 j_2\rangle \quad \text{i.e } m_1 = j_1 \text{ and } m_2 = j_2$$

state with max. projection along z-axis

- Other states with $j = j_1 + j_2$ (first column) can be obtained by lowering $|j_1 + j_2, j_1 + j_2\rangle$

$$J_- |j_1 + j_2, j_1 + j_2\rangle = \hbar\sqrt{2(j_1 + j_2)}\ |j_1 + j_2, j_1 + j_2 - 1\rangle$$

As we did earlier, $\quad J_- = J_{1-} + J_{2-}$

$$|j_1 + j_2, j_1 + j_2 - 1\rangle = \frac{1}{\hbar\sqrt{2(j_1 + j_2)}} J_- |j_1 + j_2, j_1 + j_2\rangle$$

$$= \frac{1}{\sqrt{2(j_1 + j_2)}\hbar}\left(\hbar\sqrt{2j_1}\ |j_1 (j_1 - 1), j_2 j_2\rangle + \hbar\sqrt{2j_2}\ |j_1 j_1, j_2 (j_2 - 1)\rangle\right)$$

$$= \sqrt{\frac{j_1}{j_1 + j_2}}\ |j_1 (j_1 - 1), j_2 j_2\rangle + \sqrt{\frac{j_2}{j_1 + j_2}}\ |j_1 j_1, j_2 (j_2 - 1)\rangle$$

- Keep lowering until you reach the last state in a column.

Exercise: Determine the product basis combination for top state in the 2nd column, i.e, for $|j_1 + j_2 - 1, j_1 + j_2 - 1\rangle$

Hint to the solution:

Top state in 2nd column: $|j_1+j_2-1, j_1+j_2-1\rangle = |j, m\rangle$

Product basis states has the form $|j_1 m_1, j_2 m_2\rangle$

Only those product states will contribute for which $m = m_1 + m_2$

$$\therefore \quad m = m_1 + m_2$$

$$j_1+j_2-1 = j_1 + j_2 - 1$$

OR

$$j_1 - 1 + j_2$$

m_1 cannot be j_2 or j_2-1 since $-j_1 \le m_1 \le j_1$

Hence, product states required have the form $|j_1 m_1, j_2 m_2\rangle$

i.e $|j_1 j_1, j_2(j_2-1)\rangle$ and $|j_1(j_1-1), j_2 j_2\rangle$

Answer: $|j_1+j_2-1, j_1+j_2-1\rangle = \sqrt{\dfrac{j_1}{j_1+j_2}}\, |j_1 j_1, j_2(j_2-1)\rangle - \sqrt{\dfrac{j_2}{j_1+j_2}}\, |j_1(j_1-1), j_2 j_2\rangle$

To obtain coefficients, use the conditions:

(a) normalise both kets $|j_1 j_1, j_2(j_2-1)\rangle$ and $|j_1(j_1-1), j_2 j_2\rangle$.

(b) These should be orthogonal to the other state formed out of these kets, namely, $|j_1+j_2, j_1+j_2-1\rangle$.

Sign convention: coefficient of product ket with $m_1 = j_1$ must be +ve