(1)

We know that $L_x$, $L_y$ and $L_z$ are incompatible observables. That is, they do not commute. What's the corresponding uncertainty relation?

- Uncertainty principle

$$\sigma_A^2 \, \sigma_B^2 \geq \left( \frac{1}{2i} \langle [\hat{A} \, \hat{B}] \rangle \right)^2 .$$

- Let $A = L_x$, $B = L_y$, then $[L_x \, L_y] = i\hbar L_z$

$$\therefore \quad \sigma_{L_x}^2 \, \sigma_{L_y}^2 \geq \left( \frac{1}{2i} \langle i\hbar L_z \rangle \right)^2 = \frac{\hbar^2}{4} \langle L_z \rangle^2$$

- $\sigma_{L_x} \, \sigma_{L_y} \geq \frac{\hbar}{2} |\langle L_z \rangle|$

(2)

Let $Y_2^1(\theta,\varphi) = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{i\varphi}$. Find $Y_2^2(\theta,\varphi)$.

We know $L_+ |l, m\rangle = \hbar \sqrt{(l-m)(l+m+1)} \, |l, m+1\rangle$

- $L_+ Y_2^1(\theta,\varphi) = Y_2^2(\theta,\varphi)$

- $L_+ \to \hbar e^{i\varphi} \left( \frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right)$ (in position representation)

$$\to \hbar e^{i\varphi} \left[ -\frac{2}{2} \sqrt{\frac{15}{8\pi}} \cos 2\theta \, e^{i\varphi} \cancel{+} i \cot\theta \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, i e^{i\varphi} \right]$$

$$\to -\sqrt{\frac{15}{8\pi}} \hbar e^{2i\varphi} \left( \cos 2\theta - \cos^2\theta \right) \to \sqrt{\frac{15}{8\pi}} \hbar e^{2i\varphi} \sin^2\theta$$

Now, normalise the state using \theta and \phi dependence. This is done in next page.

(2)

$$\int_0^\infty |R|^2 r^2\, dr = 1 \qquad \int_0^{2\pi}\int_0^{\pi} |Y|^2 \sin\theta\, d\theta\, d\varphi = 1$$

$$c^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^4\theta \sin\theta\, d\theta = 1 \quad \Rightarrow \quad c = \sqrt{\frac{15}{32\pi}}$$

- $$Y_2^2(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\, \sin^2\theta\, e^{2i\varphi}$$

(3) Hydrogen atom is in a superposition state

$$\psi(r,0) = \frac{1}{\sqrt{2}}\left[\psi_{2,1,1} + \psi_{2,1,-1}\right]$$

Find $\psi(r,t)$.

- $$\psi(r,t) = e^{-i\hat{H}t/\hbar}\,\psi(r,0)$$
$$= e^{-iHt/\hbar}\,\frac{1}{\sqrt{2}}\left[\psi_{2,1,1} + \psi_{2,1,-1}\right]$$

$$H\,\psi_{2,1,1} = E_2\,\psi_{2,1,1} \quad \text{where} \quad E_2 = \frac{-\mu e^4}{2\hbar^2 4}$$

$\therefore$

- $$\psi(r,t) = \frac{1}{\sqrt{2}}\, e^{-iE_2 t/\hbar}\left[\psi_{2,1,1} + \psi_{2,1,-1}\right]$$

Simplify further by using explicit forms for $\psi_{2,1,1}$ & $\psi_{3,1,-1}$.

(4)
Electron is in spin state

$\chi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}$

Find Normalisation constant.

$|A|^2 (-3i \quad 4) \begin{pmatrix} 3i \\ 4 \end{pmatrix} = 1$

$|A|^2 \, 25 = 1 \Rightarrow A = 1/5. \qquad \chi = \frac{1}{5} \begin{pmatrix} 3i \\ 4 \end{pmatrix}$

Find expectation value of $S_x$.

We need $\langle \chi | S_x | \chi \rangle =$

$A^* (-3i \quad 4) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 3i \\ 4 \end{pmatrix} = 0. \qquad \langle S_x \rangle = 0$

Find $\sigma_{S_x}^2 = (S_x - \langle S_x \rangle)^2 = S_x^2$ ? So, we need $\langle \chi | S_x^2 | \chi \rangle$

Since $S_x = \frac{\hbar}{2} \sigma_x$ and $S_x^2 = \frac{\hbar^2}{4} \sigma_x^2 = \frac{\hbar^2}{4} I$

$\sigma_{S_x}^2 = A^* (-3i \quad 4) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\hbar^2}{4} \begin{pmatrix} 3i \\ 4 \end{pmatrix} A = \frac{|A|^2 \hbar^2 \, 25}{4}$

$\sigma_{S_x}^2 = \frac{\hbar^2}{4}$

(5) A Spin-1/2 Particle is in state $\chi = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$. 

If we measure $S_z$ and $S_x$, find probability of getting $+\hbar/2$ and $-\hbar/2$.

Consider $S_z$ operator first.

- Eigenvectors of $S_z$: $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Prob. of getting $|\uparrow\rangle$: $|\langle\uparrow|\chi\rangle|^2$

Note $|\chi\rangle = \frac{1+i}{\sqrt{6}}|\uparrow\rangle + \frac{2}{\sqrt{6}}|\downarrow\rangle$

we need: $\langle\uparrow|\chi\rangle = \langle\uparrow|\left(\frac{1+i}{\sqrt{6}}|\uparrow\rangle + \frac{2}{\sqrt{6}}|\downarrow\rangle\right)$

$= \frac{1+i}{\sqrt{6}}$ since $\langle\uparrow|\downarrow\rangle = 0$

- $\therefore |\langle\uparrow|\chi\rangle|^2 = \frac{(1+i)(1-i)}{6} = \frac{2}{6} = \frac{1}{3}$

- Similarly, $|\langle\downarrow|\chi\rangle|^2 = \frac{2}{3}$.

Now, consider $S_x$ operator:

- Eigenvectors of $S_x$: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Let $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$

$|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}}$

Prob. of getting $|\uparrow\rangle$: $|\langle +|x\rangle|^2$

$$\langle +|x\rangle = \left(\frac{\langle\uparrow| + \langle\downarrow|}{\sqrt{2}}\right)\left(\frac{1+i}{\sqrt{6}}|\uparrow\rangle + \frac{2}{\sqrt{6}}|\downarrow\rangle\right) = \frac{1+i}{\sqrt{12}} + \frac{2}{\sqrt{12}} = \frac{3+i}{\sqrt{12}}$$

- $|\langle +|x\rangle|^2 = \frac{|3+i|^2}{12} = \frac{10}{12} = \frac{5}{6}$

- Similarly, $|\langle -|x\rangle|^2 = \frac{1}{6}$.

(6) Express $\sqrt{I + i\sigma_x}$ in terms of Pauli matrices.

- $(I + i\sigma_x)^{1/2} = \left[\beta\left(I \cos\frac{\theta}{2} - i \sin\frac{\theta}{2}\, \hat{\theta}\cdot\vec{\sigma}\right)\right]^{1/2}$

Since second term on LHS involves only $\sigma_x$, let us put $\hat{\theta} = \hat{i}$.

$\therefore$ we need to satisfy $\beta \cos\frac{\theta}{2} = 1$ and $-\beta \sin\frac{\theta}{2} = 1$

we have $\theta = -\frac{\pi}{2}$. So that $\beta \cdot \frac{1}{\sqrt{2}} = 1$ & $-\beta\left(-\frac{1}{\sqrt{2}}\right) = 1$

$\therefore$ $\beta = \sqrt{2}$.

Then $(I + i\sigma_x)^{1/2} = \left(\sqrt{2}\, e^{-i(-\pi/4)\sigma_x}\right)^{1/2}$

$= 2^{1/4} e^{i\pi/8\, \sigma_x} = 2^{1/4}\left(\cos\frac{\pi}{8} + i \sin\frac{\pi}{8}\sigma_x\right)$

$$\therefore (I + i\sigma_x)^{1/2} = 2^{1/4} \cos\frac{\pi}{8}\, I + 2^{1/4} i \sin\frac{\pi}{8}\, \sigma_x$$