A particle is described by a wavefunction

$$\psi(\rho, \varphi) = A\, e^{-\rho^2/2\Delta^2} \cos^2\varphi$$

show that

$$P(l_z = 0) = \frac{2}{3}, \qquad P(l_z = 2\hbar) = \frac{1}{6}, \qquad P(l_z = -2\hbar) = \frac{1}{6}.$$

Let $l_z = m\hbar$ and $\Phi_m(\varphi) = \dfrac{e^{im\varphi}}{\sqrt{2\pi}} = |\Phi(\varphi)\rangle$

(See notes on eigenfunctions of L_z)

Also, note that $\cos^2\varphi = \dfrac{1 + \cos 2\varphi}{2}$

$$\therefore \cos^2\varphi = \left(\frac{1}{2} + \frac{e^{2i\varphi} + e^{-2i\varphi}}{4}\right)\frac{\sqrt{2\pi}}{\sqrt{2\pi}}$$

$$\cos^2\varphi = \frac{\sqrt{2\pi}}{2}\left[|\Phi_0(\varphi)\rangle + \frac{|\Phi_2(\varphi)\rangle}{2} + \frac{|\Phi_{-2}(\varphi)\rangle}{2}\right] \quad \text{--- (1)}$$

Required probability is

$$P(m) = |\langle \Phi_m(\varphi) | \psi(\rho, \varphi)\rangle|^2$$

Calculate *P(0)*

$$\langle \Phi_0(\varphi) | \psi(\rho,\varphi)\rangle = \int_0^\infty A\, e^{-\rho^2/2\Delta^2} \rho\, d\rho \int_0^{2\pi} \Phi_0^*(\varphi) \cos^2\varphi\, d\varphi$$

Note: Area element in 2D cylindrical coordinate system is $\rho\, d\rho\, d\varphi$.
(Show this yourself).

Let $I_\rho = \int_0^\infty A\, e^{-\rho^2/2\Delta^2}\, \rho\, d\rho$

$$\therefore \langle \Phi_m(\varphi) | \psi(\rho,\varphi) \rangle = I_\rho \int_0^{2\pi} \Phi_m^*(\varphi) \cos^2\varphi \, d\varphi$$

Now substitute for $\cos^2\varphi$ from Eq.(1).

Then,

$$\langle \Phi_m(\varphi) | \psi(\rho,\varphi) \rangle = I_\rho \sqrt{\frac{\pi}{2}} \Big[\langle \Phi_m(\varphi) | \Phi_0(\varphi) \rangle + \frac{1}{2} \langle \Phi_m(\varphi) | \Phi_2(\varphi) \rangle + \frac{1}{2} \langle \Phi_m(\varphi) | \Phi_{-2}(\varphi) \rangle \Big]$$

--------(2)

Note that $\langle \Phi_m(\varphi) | \Phi_{m'}(\varphi) \rangle = \delta_{m,m'}$

$\therefore$ From Eq (2), we get

$$\langle \Phi_0(\varphi) | \psi(\rho,\varphi) \rangle = I_\rho \sqrt{\frac{\pi}{2}}$$

$$\langle \Phi_2(\varphi) | \psi(\rho,\varphi) \rangle = \langle \Phi_{-2}(\varphi) | \psi(\rho,\varphi) \rangle = \frac{I_\rho}{2} \sqrt{\frac{\pi}{2}}$$

$$P(0) = I_\rho^2 \frac{\pi}{2}, \qquad P(2) = \frac{I_\rho^2}{4} \frac{\pi}{2}, \qquad P(-2) = \frac{I_\rho^2}{4} \frac{\pi}{2}$$

Probabilities are in the ratio

$$P(0) : P(2) : P(-2)$$

$$1 : \frac{1}{4} : \frac{1}{4}$$

Since, we must satisfy $c\left[P(0) + P(2) + P(-2)\right] = 1$,

where c is normalisation constant.

$$c\left(\frac{3}{2}\right) = 1 \Rightarrow c = \frac{2}{3}$$

Hence, normalised probability is

$$P(0) : P(2) : P(-2)$$

$$\frac{2}{3} : \frac{1}{6} : \frac{1}{6}$$

Note that I_ρ need not be evaluated.
This is because dynamics in ρ direction is
decoupled from dynamics in φ direction.
This may not be always true, but is true
in this problem (and for most problems
we might do in this course).