PH 3124 Brief solutions. Quiz-2

1. $\mathbb{P}^2 \psi(x) = \psi(x)$. Hence, eigenvalues are ± 1.

$$\mathbb{P}\left(e^{-x^2/2} \cos x\right) = e^{-(-x)^2/2} \cos(-x)$$
$$= e^{-x^2/2} \cos x$$

$\therefore$ Eigenvalue is $+1$. Hence, even parity.

2. $E = 2k + |m| + 1$, $\quad k = 0, 1, 2, \ldots$

If $n = 2k + |m|$

For a given n, $k = 0, 1, 2, \ldots n-1$.

Degeneracy is $n+1$.

3. Let $H\psi(x) = E\psi(x)$. Note $\mathbb{P}\psi(x) = \psi(-x)$

$$\mathbb{P} H \psi(x) = E\, \mathbb{P} \psi(x)$$

Since H and $\mathbb{P}$ commute, we have

$$H\, \mathbb{P} \psi(x) = E\, \mathbb{P} \psi(x)$$

$$\therefore \quad H \psi(-x) = E \psi(-x)$$

Thus, $\psi(-x)$ is also an eigenstate with same eigenvalue.

4. $$\delta(x^2 - a^2) = \frac{1}{2a}\left[\delta(x-a) + \delta(x+a)\right]$$

$$\int_{-\infty}^{\infty} \delta(x^2-a^2)\, f(x)\, dx = \int_{-\infty}^{0} \delta(x^2-a^2)\, f(x)\, dx + \int_{0}^{\infty} \delta(x^2-a^2)\, f(x)\, dx \quad \text{---} (1)$$

$f(x) \to$ some probe function

put $x = -\sqrt{u}$, $\quad dx = \dfrac{-du}{2\sqrt{u}}$

$\therefore x^2 = u$

Then, $$\int_{-\infty}^{0} \delta(x^2-a^2)\, f(x)\, dx = -\frac{1}{2}\int_{\infty}^{0} \delta(u-a^2)\, \frac{f(-\sqrt{u})}{\sqrt{u}}\, du$$

$$= \frac{1}{2}\int_{0}^{\infty} \delta(u-a^2)\, \frac{f(-\sqrt{u})}{2\sqrt{u}}\, du = \frac{1}{2a} f(-a)$$

We have used the identity $\int g(y)\, \delta(y-\alpha)\, dy = g(\alpha)$

Consider the 2nd term in Eqn (1) : $\int_{0}^{\infty} \delta(x^2-a^2)\, f(x)\, dx$

put $x = \sqrt{u}$, $\quad dx = \dfrac{du}{2\sqrt{u}}$

$x^2 = u$

$$\int_{0}^{\infty} \delta(x^2-a^2)\, f(x)\, dx = \int_{0}^{\infty} \delta(u-a^2)\, \frac{f(\sqrt{u})}{2\sqrt{u}}\, du$$

$$= \frac{1}{2a} f(a)$$

$\therefore$

$$\int_{-\infty}^{\infty} \delta(x^2-a^2)\, f(x)\, dx = \frac{1}{2a}\left[f(-a) + f(a) \right]$$

note that $f(a) = \int f(x)\, \delta(x-a)\, dx$

$$\int_{0}^{\infty} \delta(x^2-a^2)\, f(x)\, dx = \frac{1}{2a}\left[\int f(x)\, \delta(x+a)\, dx + \int f(x)\, \delta(x-a)\, dx \right]$$

In "unofficial" notation:

$$\delta(x^2-a^2) = \frac{1}{2a}\left[\delta(x+a) + \delta(x-a) \right]$$

5. $U^{\dagger}[R(\varepsilon)]\; P_y\; U[R(\varepsilon)] = \left(I + \frac{i\varepsilon}{\hbar} L_z\right) P_y \left(I - \frac{i\varepsilon}{\hbar} L_z\right)$

$$= P_y + \frac{i\varepsilon}{\hbar}\left(L_z P_y - P_y L_z\right) = P_y + \frac{i\varepsilon}{\hbar}\left[L_z\, P_y\right]$$

$$= P_y + \frac{i\varepsilon}{\hbar}\left[L_z, P_y\right]$$

$$= P_y - \frac{i\varepsilon}{\hbar}\left[P_y, L_z\right] \qquad \text{-----} \; 2$$

$$= P_y - \frac{i\varepsilon}{\hbar}\left[P_y (X P_y - Y P_x) - (X P_y - Y P_x) P_y \right]$$

$$= P_y - \frac{i\varepsilon}{\hbar}\left(\cancel{P_y X P_y} - P_y Y P_x - \cancel{X P_y P_y} + Y P_x P_y \right)$$

- Used the fact that $P_x P_y = P_y P_x$,
- and $[x, P_y] = 0$. Hence, the cancellation.

We are left with

$$= P_y - \frac{i\varepsilon}{\hbar}\left(y P_x P_y - P_y y P_x \right)$$

$$= P_y - \frac{i\varepsilon}{\hbar}\left(y P_y - P_y y \right) P_x$$

$$= P_y - \frac{i\varepsilon}{\hbar}[y, P_y] P_x = P_y - \frac{i\varepsilon}{\hbar}(i\hbar) P_x$$

$$= P_y + \varepsilon P_x \qquad \text{------} \quad (3)$$

Comparing Eq(2) and Eq(3), we get

$$\frac{-i}{\hbar}[P_y, L_z] = P_x$$

This gives $[P_y, L_z] = i\hbar P_x$.