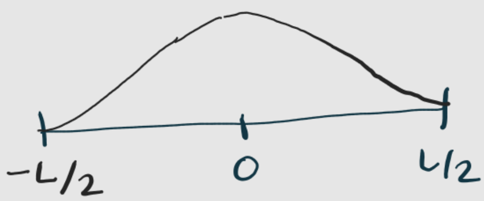
1) Dimension of $\hbar$: Energy x time $\equiv ML^2T^{-2} \times T = ML^2T^{-1}$

2a) $\langle \hat{O}_1 \varphi | \psi \rangle$

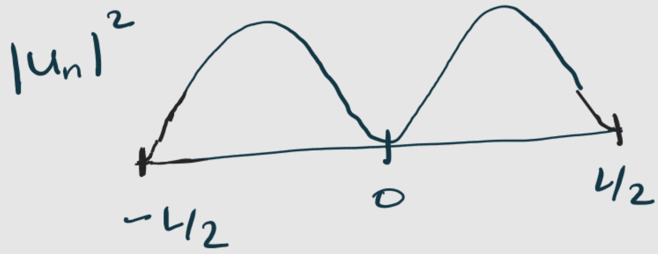
(2b) $\langle \varphi | \varphi \rangle$

3)

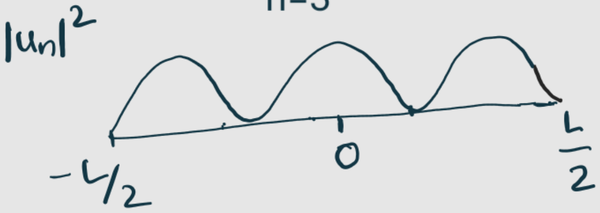
n=1



n=2



n=3



$|u_n|^2 = u_n^* u_n$

4) $$[P^2, X] = PPX - XPP$$

$$= PPX - XPP + \underbrace{PXP - PXP}_{\text{added and subracted}}$$

$$= P(PX - XP) - (XP - PX)P$$

$$= -P(XP - PX) - (XP - PX)P$$

$$= -P[X, P] - [X, P]P$$

$$= -2P[X, P] = -2Pi\hbar = -2i\hbar P$$

$$\therefore [P^2, X] = -2i\hbar P$$

we have used $[X, P] = i\hbar$

4) $\psi(x) = A\, e^{ikx}\, e^{-\alpha x}$

calculating current $\quad j = \frac{\hbar}{2mi}\left(\psi^* \psi' - \psi\, \psi^{*\prime}\right)$

$$\psi^* = A\, e^{-ikx}\, e^{-\alpha x}$$

$$\psi' = A\, e^{ikx}\, e^{-\alpha x}\,(ik - \alpha)$$

$$\psi^{*\prime} = A\, e^{-ikx}\, e^{-\alpha x}\,(-ik - \alpha)$$

substitute these and simplify to get

$$j = \frac{\hbar k}{m} |A|^2\, e^{-2\alpha x}$$

5) Given that $\psi(x) = 1/L \quad \left(-\frac{L}{2} \leq x < \frac{L}{2}\right)$

Ground state of infinite well: $\quad \varphi_1(x) = \frac{1}{\sqrt{L}} \cos\left(\frac{\pi x}{L}\right)$

NOTE : The normalisation factor is incorrect in the question paper.
It should have been $2/\sqrt{L}$.

$\therefore$ Probability that $\psi(x)$ is in ground state is

$$|\langle \varphi_1 | \psi \rangle|^2 = \left| \int \varphi_1^*(x)\, \psi(x)\, dx \right|^2$$

$$\text{Let} \quad I = \int_{-L/2}^{L/2} \varphi_1(x)\, \psi(x)\, dx$$

$$I = \frac{1}{L^{3/2}} \int_{-L/2}^{L/2} \cos\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{1}{L^{3/2}} \frac{L}{\pi} \sin\left(\frac{\pi x}{L}\right) \Big|_{-L/2}^{L/2}$$

$$= \frac{1}{\pi\sqrt{L}} \cdot 2 = \frac{2}{\pi\sqrt{L}}$$

$$\therefore \ |\langle \varphi_1 | \psi \rangle|^2 = I^2 = \left(\frac{2}{\pi\sqrt{L}}\right)^2 = \frac{4}{\pi^2 L}$$

- If the normalisation factor $\sqrt{\frac{2}{L}}$ was used, the answer would be : $8/\pi^2 L$.