Quantum Mechanics 1 | Brief solutions for midsem exam

1. $$\Psi(x,t) = A\, e^{-a\left(\frac{mx^2}{\hbar} + it\right)}$$

To find A : $$\int_{-\infty}^{\infty} \Psi^*(x,t)\, \Psi(x,t)\, dx = 1$$

This gives $$A^2 \int_{-\infty}^{\infty} e^{-2amx^2/\hbar}\, dx = 1$$

$$A^2 \sqrt{\frac{\pi\hbar}{2am}} = 1 \Rightarrow A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}$$

2. De Broglie wavelength $\lambda_D = \frac{h}{p}$

Note $E = \frac{p^2}{2m}$. $\therefore p = \sqrt{2mE}$. $\therefore \lambda_D = \frac{h}{\sqrt{2mE}}$

3. $$\hat{P} = i\sqrt{\frac{m\omega\hbar}{2}}\,(a^+ - a)\,(a^+ - a)$$

$$\hat{P}^2 = -\left(\frac{m\omega\hbar}{2}\right)\left(a^{+2} - a^+ a - a a^+ + a^2\right)$$

$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $a|n\rangle = \sqrt{n}\,|n-1\rangle$

$$a^{+2}|n\rangle = \sqrt{(n+1)(n+2)}\,|n+2\rangle$$

$$a^2|n\rangle = \sqrt{n(n-1)}\,|n-2\rangle$$

$$a^+a|n\rangle = n|n\rangle$$

$$aa^+|n\rangle = (n+1)|n\rangle$$

$$E_{avg} = \frac{1}{2m}\left(-\frac{m\omega\hbar}{2}\right) \langle n | a^{+2} - a^{+}a - aa^{+} + a^2 | n \rangle$$

$$= -\frac{\hbar\omega}{4}\Big[\sqrt{(n+1)(n+2)} \; \langle n | n+2 \rangle - n \langle n | n \rangle$$

$$- (n+1) \langle n | n \rangle + \sqrt{n(n-1)} \; \langle n | n-2 \rangle \Big]$$

$$E_{avg} = \frac{\hbar\omega}{4}(2n+1) - \sqrt{(n+1)(n+2)}\, \delta_{n,n+2} - \sqrt{n(n-1)}\, \delta_{n,n-2}$$

4. $[\hat{p}, \hat{x} \sin\hat{p}] = [\hat{p}, \hat{x}] \sin\hat{p} - \hat{x}[\hat{p}, \sin\hat{p}]$

$$= -i\hbar \sin\hat{p} - 0$$

$$= -i\hbar \sin\hat{p}$$

5. Let $\hat{O}|\varphi_1\rangle = \lambda_1 |\varphi_1\rangle$ and $\hat{O}|\varphi_2\rangle = \lambda_2 |\varphi_2\rangle$

$$\Downarrow$$

$\langle \varphi_2 | \hat{O} | \varphi_1 \rangle = \lambda_1 \langle \varphi_2 | \varphi_1 \rangle$ — ①

$$\langle \varphi_1 | \hat{O} | \varphi_2 \rangle = \lambda_2 \langle \varphi_1 | \varphi_2 \rangle$$

$$\langle \varphi_1 | \hat{O} | \varphi_2 \rangle^{+} = \lambda_2 \langle \varphi_1 | \varphi_2 \rangle^{+}$$

$$\langle \varphi_2 | \hat{O}^{+} | \varphi_1 \rangle = \lambda_2 \langle \varphi_2 | \varphi_1 \rangle$$ — ②

Eq. ① − ② $= (\lambda_1 - \lambda_2) \langle \varphi_2 | \varphi_1 \rangle = 0$

$\therefore \langle \varphi_2 | \varphi_1 \rangle = 0$ (Note that $\hat{O}^{+} = \hat{O}$)

6. Given that $\langle p \rangle = \langle \psi | \hat{p} | \psi \rangle$

$$\langle p \rangle = \int \psi^* \left(-i\hbar \frac{d}{dx} \right) \psi \; dx$$

Using $\varphi(x) = e^{ip_0 x/\hbar} \psi(x)$

$$= \int e^{-ip_0 x/\hbar} \psi^*(x) \left(-i\hbar \frac{d}{dx} \right) e^{ip_0 x/\hbar} \psi(x) \; dx$$

$$= (-i\hbar) \int e^{-ip_0 x/\hbar} \psi^*(x) \left[\frac{ip_0}{\hbar} e^{ip_0 x/\hbar} \psi(x) + e^{ip_0 x/\hbar} \psi'(x) \right] dx$$

$$= (-i\hbar) \int \varphi^*(x) \, \varphi(x) \frac{ip_0}{\hbar} + \psi^*(x) \frac{d\psi}{dx} \Big] \; dx$$

$$= p_0 + \int \psi^*(x) \left(-i\hbar \frac{d}{dx} \right) \psi(x) \; dx$$

$$= p_0 + \langle p \rangle.$$

7. Erhenfest theorem: $\frac{d}{dt} \langle \hat{A} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{A}] \rangle$

$\hat{A} = \hat{x}^2$, and taking $\hat{H} = \frac{p^2}{2m} + V(x)$

$$\frac{d}{dt} \langle \hat{x}^2 \rangle = \frac{i}{\hbar} \langle [H, \hat{x}^2] \rangle$$

$$\left[\frac{\hat{P}^2}{2m} + V(x), \hat{X}^2\right] = \left[\frac{\hat{P}^2}{2m}, \hat{X}^2\right] + \left[V(\hat{x}), \hat{X}^2\right]$$

$$= \frac{1}{2m}\left[\hat{P}^2, \hat{X}^2\right] + 0$$

$$= \frac{1}{2m}\left(\left[\hat{P}^2, \hat{x}\right]\hat{x} + \hat{x}\left[\hat{P}^2, \hat{x}\right]\right)$$

Note $\left[\hat{P}^2, \hat{x}\right] = -2i\hbar\hat{P}$

$$\left[\frac{\hat{P}^2}{2m} + \hat{V}(\hat{x}), \hat{X}^2\right] = -\frac{2i\hbar}{2m}\left[\hat{P}\hat{X} + \hat{x}\hat{P}\right]$$

$$\left[\hat{H}, \hat{x}^2\right] = \frac{\hbar}{im}\left[\hat{P}\hat{X} + \hat{x}\hat{P}\right]$$

$$\therefore \frac{d}{dt}\langle\hat{x}^2\rangle = \frac{i}{\hbar}\left\langle\frac{\hbar}{im}\left(\hat{p}\hat{x} + \hat{x}\hat{p}\right)\right\rangle$$

$$m\frac{d}{dt}\langle\hat{x}^2\rangle = \langle\hat{P}\hat{X} + \hat{x}\hat{P}\rangle = \langle\hat{P}\hat{x}\rangle + \langle\hat{x}\hat{P}\rangle.$$

8. (a) $\hat{A}\hat{B} - \hat{B}\hat{A} = 0$

Let $A\varphi = a\varphi$

$$BA\varphi = aB\varphi$$

$$A(B\varphi) = a(B\varphi) \quad \text{Since } [A, B] = 0$$

$B\varphi$ is also an eigenfunction of A with eigenvalue a.

Hence, $B\varphi$ and φ must be related by a constant: $B\varphi = b\varphi$.

Hence, $\hat{A}$ and $\hat{B}$ share a common eigenstate φ.

(b) $[A, B] = iC \implies AB - BA = iC$

$$C = \frac{AB - BA}{i}$$

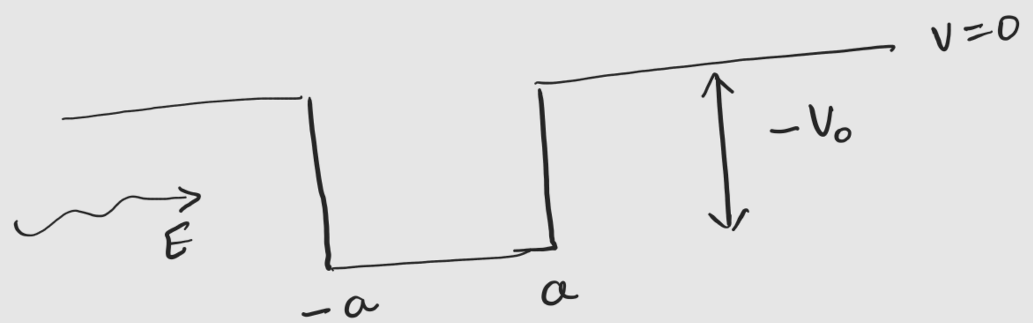
$$C^{+} = \frac{(AB - BA)^{+}}{-i} = \frac{(AB)^{+} - (BA)^{+}}{-i}$$

Since A & B are Hermitian, $A^{+} = A$ and $B^{+} = B$.

$$C^{+} = \frac{BA - AB}{-i} = \frac{AB - BA}{i} = C$$

$\therefore$ C is Hermitian.

9.



Incoming wavepacket only from left with energy E

Schrodinger equation

$$(|x| > a) \qquad \frac{d^2\psi_1}{dx^2} - \kappa^2 \psi_1 = 0, \qquad \kappa = \sqrt{\frac{-2mE}{\hbar}} \, .$$

$$(|x| < a) \qquad \frac{d^2\psi_2}{dx^2} + q^2 \psi = 0, \qquad q = \sqrt{\frac{2m(E+V_0)}{\hbar}}$$

Since we need only even (parity) states, the required solution is inside the well,

$$\psi_2(x) = A \cos qx$$

For $|x| > a$, we expect decaying solutions

$$\psi_1(x) = e^{\kappa x} \qquad (x < a)$$
$$= e^{-\kappa x} \qquad (x > a)$$

Applying boundary conditions

$$\psi_1(-a) = \psi_2(-a), \qquad \text{and} \qquad \psi_1'(-a) = \psi_2'(-a)$$

$$e^{-\kappa a} = A \cos qa \qquad \kappa e^{-\kappa a} = A q \sin qa$$

Dividing the last two equations, we get

$$\frac{Aq \sin qa}{A \cos qa} = \frac{k\, e^{-ka}}{e^{-ka}}$$

$$\Rightarrow \quad q \tan qa = k \quad \Rightarrow \quad \tan qa = \left(k/q\right)$$

10. Let $u(x,0) = \frac{3}{5}\psi_0(x) + \frac{4}{5}\psi_1(x)$

$u(x,0)$ is already normalised.

(a)

$$u(x,t) = e^{-i\hat{H}t/\hbar}\, u(x,0)$$

$$= e^{-i\hat{H}t/\hbar}\left[\frac{3}{5}\psi_0(x) + \frac{4}{5}\psi_1(x)\right]$$

Note : $\hat{H}\psi_0 = E_0\psi_0 = \frac{\hbar\omega}{2}\psi_0$ and $\hat{H}\psi_1 = E_1\psi_1 = \frac{3\hbar\omega}{2}\psi_1$

$$u(x,t) = \frac{3}{5} e^{-i\frac{\hbar\omega}{2\hbar}t}\psi_0(x) + \frac{4}{5} e^{-i\frac{3\hbar\omega}{2\hbar}t}\psi_1(x)$$

$$= \underbrace{\frac{3}{5} e^{-i\frac{\omega}{2}t}}_{c_0}\psi_0(x) + \underbrace{\frac{4}{5} e^{-i\frac{3\omega}{2}t}}_{c_1}\psi_1(x)$$

$$u(x,t) = c_0\,\psi_0(x,t) + c_1\,\psi_1(x)$$

(b) $\langle x \rangle = \langle u(x,t) | \hat{x} | u(x,t) \rangle$

$$\langle x \rangle = \int_{-\infty}^{\infty} u^*(x,t)\ \hat{x}\ u(x,t)\ dx$$

Let's write $u(x,t) = c_0 \psi_0 + c_1 \psi_1$

where $c_0 = \frac{3}{5} e^{-i\frac{\omega}{2}t}$

Then, $u^*(x,t) = c_0^* \psi_0 + c_1^* \psi_1$ (ψ_0 and ψ_1 are real).

$$\langle x \rangle = \int_{-\infty}^{\infty} (c_0^* \psi_0 + c_1^* \psi_1)\ \hat{x}\ (c_0 \psi_0 + c_1 \psi_1)\ dx$$

$$= \int_{-\infty}^{\infty} \left[|c_0|^2 \psi_0^2 + |c_1|^2 \psi_1^2 + \psi_0 \psi_1 \left(c_0^* c_1 + c_0 c_1^* \right) \right] x\ dx$$

$\psi_0^2 x$ and $\psi_1^2 x$ are both odd function.

They would vanish under the integral.

Then, $\langle x \rangle = \int_{-\infty}^{\infty} x\ \psi_0 \psi_1\ 2\ \mathrm{Re}(c_0^* c_1)\ dx$

$$\langle x \rangle = 2\ \mathrm{Re}(c_0^* c_1) \int_{-\infty}^{\infty} x\ \psi_0 \psi_1\ dx$$

$$\mathrm{Re}\,(c_0^* c_1) = \mathrm{Re}\left(\frac{3}{5} e^{iE_0 t/\hbar} \quad \frac{4}{5} e^{-iE_1 t/\hbar} \right)$$

$$= \frac{12}{25} \mathrm{Re}\left(e^{i(E_0 - E_1)t/\hbar} \right)$$

$$= \frac{12}{25} \mathrm{Re}\left(e^{\frac{i}{\hbar}\left(\frac{\hbar\omega}{2} - \frac{3}{2}\hbar\omega\right)t} \right)$$

$$= \frac{12}{25} \mathrm{Re}\left(e^{-i\omega t} \right) = \frac{12}{25} \cos \omega t$$

Hence:

$$\langle x \rangle = \frac{24}{25} \underbrace{\int_{-\infty}^{\infty} x \, \psi_0(x) \, \psi_1(x) \, dx}_{I} \quad \cos \omega t$$

$$\therefore \quad \langle x \rangle = \frac{24}{25} \cos(\omega t) \, I$$

This shows that $\langle x \rangle$ oscillates with frequency ω.