Indian Institute of Science Education and Research Pune
Mid-semester Exam, Aug (2024) semester.

Course name: Quantum Mechanics I — **Course code:** PH-3124
Date: 19.9.2024 (10:00 AM to 12:00 PM) — *Duration:* 2 hours
Instructor : M. S. Santhanam — Maximum marks: 60

- Among questions 1 to 6, answer ANY FIVE of them. Questions 7 to 10 are compulsory.
- If you draw sketches as an answer, label the axes. No marks without axes labels.
- SHOW ALL THE STEPS CLEARLY in your calculations while arriving at an answer.
- Unless specified otherwise, all the symbols have their usual meanings.
- Use the same symbols and notation given in the question. Do not use your own.

1. A particle of mass m with parameter $a > 0$ is in the state

$$\psi(x,t) = A\ e^{-a\left((mx^2/\hbar)+it\right)}, \quad -\infty \leq x \leq \infty.$$

Find A. (5)

2. For an electron of mass m with kinetic energy E, show that its de Broglie wavelength is $\lambda_D = \frac{h}{2mE}$, where h is Planck constant. (5)

3. For a harmonic oscillator with frequency ω_0 and mass m, compute the average kinetic energy given by

$$E_{\text{avg}} = \left(\frac{1}{2m}\right) \langle n|\widehat{P}^2|n\rangle,$$

where $|n\rangle$ represents harmonic oscillator eigenstates. Write the answer in its most simplified form. (5)

4. Find the value of the commutator $[\widehat{p}, \widehat{x}\ \sin(\widehat{p})]$. (5)

5. For a Hermitian operator $\widehat{O}$, show that eigenstates belonging to two different eigenvalues are orthogonal. (5)

6. Show that if the state $\psi(x)$ has mean momentum $\langle P\rangle$, then the state $e^{ip_0x}\psi(x)$ has mean momentum $\langle P\rangle + p_0$. (5)

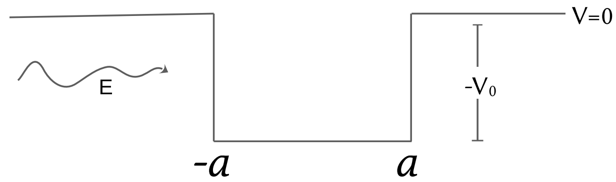
7. For Hermitian operators x and p, use the Ehrenfest theorem to show that

$$m\frac{d}{dt}\langle x^2\rangle = \frac{1}{i\hbar}\left(\langle xp\rangle + \langle px\rangle\right)$$

(8)

8. If A and B are two Hermitian operators, then answer the following:
(a) If $[A, B] = 0$, then show that operators A and B share a common eigenstate.
(b) If $[A, B] = iC$, show that the operator C is Hermitian. (4+4)

9. Consider the potential well shown below in the figure.



If E is the energy of the particle lying in the range, $-V_0 \leq E \leq 0$, then show that the quantisation condition for even states is

$$\tan qa = \kappa/q,$$

where $\kappa = \sqrt{-2mE}/\hbar$ and $q = \sqrt{2m(E+V_0)}/\hbar$. (9)

10. A particle is in a oscillator potential and has the initial state at time $t = 0$:

$$u(x, t = 0) = \frac{3}{5}\psi_0(x) + \frac{4}{5}\psi_1(x)$$

In this, $\psi_0(x)$ and $\psi_1(x)$ are the ground state and first excited state of harmonic oscillator with energies, respectively, $E_0 = \hbar\omega/2$ and $E_1 = 3\hbar\omega/2$.
(a) Find $u(x, t)$ at some other time t.
(b) Find $\langle x \rangle$ at time t and show that it oscillates with frequency ω. (5+5)

Some useful information. You can use this information in your calculation (if neeeded):

1) Unit step function definition :
$\Theta(x) = 1$ if $(x > 0)$, AND $\Theta(x) = 0$, if $x < 0$.
The potential well in the figure can be mathematically written as $V(x) = -V_0\, \Theta(a - |x|)$, with $V_0 > 0$.

2) Harmonic oscillator eigenstates $\psi_n(x)$:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right),$$

where $H_n(y)$ are Hermite polynomials. First two Hermite polynomials are :

$$H_0(y) = 1$$

$$H_1(y) = 2y$$

3) A useful integral involving harmonic oscillator eigenstates $\psi_0(x)$ and $\psi_1(x)$:

$$\int_{-\infty}^{\infty} \psi_0^*(x)\ x\ \psi_1(x)\ dx = I$$