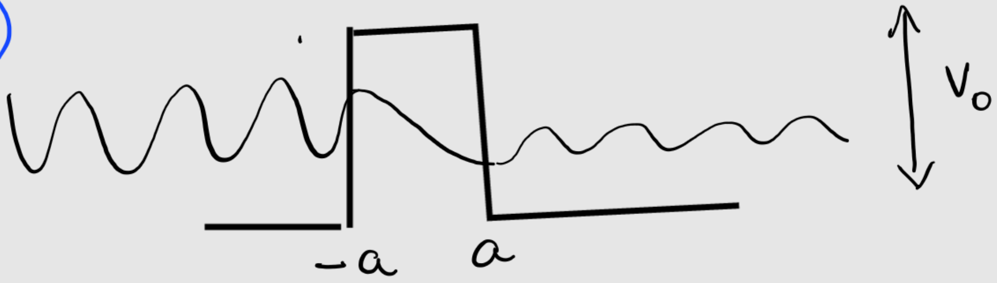
PH3124 End-semester exam Nov 2024

1a)



b) In the limit $V_0 \to \infty$, $R = 1$ and $T = 0$.

2a) conserved quantities

Energy E

Total angular momentum squared L^2

z-component of angular momentum L_z

Commutation relations:

$$[H, H] = 0, \quad [H, L^2] = 0, \quad [H, L_z] = 0.$$

b) $m = -l, -l+1 \ldots 0, 1, 2, \ldots l$

degree of degeneracy : $2l+1$

3)
$$\langle x \rangle = \int_0^\infty |A|^2 \, x \, e^{-|x|/\Delta} \, dx$$

$$\langle x \rangle = |A|^2 \Delta^2 \int_0^\infty \frac{x}{\Delta} \, e^{-x/\Delta} \, \frac{dx}{\Delta}$$

put $u = x/\Delta$

$$\langle x\rangle = |A|^2 \Delta^2 \underbrace{\int_0^\infty u\, e^{-u}\, du}_{1}$$

$$\therefore \langle x\rangle = |A|^2 \Delta^2$$

4) $[L_x, L_y]_+ = L_x L_y + L_y L_x$

put $L_x = \frac{1}{2}(L_+ + L_-)$, $\quad L_y = \frac{1}{2i}(L_+ - L_-)$

Now,

$$L_x L_y + L_y L_x = \frac{1}{4i}\Big\{(L_+ + L_-)(L_+ - L_-) + (L_+ - L_-)(L_+ + L_-)\Big\}$$

$$= \frac{1}{4i}\left\{L_+^2 - L_-^2 + L_+^2 - L_-^2\right\}$$

$$= \frac{1}{2i}\left(L_+^2 - L_-^2\right) = -\frac{i}{2}\left(L_+^2 - L_-^2\right)$$

5) $A = \begin{pmatrix} 1 & 3 \\ 5 & 4 \end{pmatrix}$

$$A = m_0 I + m_1 \sigma_1 + m_2 \sigma_2 + m_3 \sigma_3$$

$$m_i = \frac{1}{2} \operatorname{Tr}(A\sigma_i)$$

Using this, we get $m_0 = \frac{5}{2}$, $m_1 = 4$, $m_2 = -i$, $m_3 = \frac{-3}{2}$

6) $\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\,(a^{+} - a)$

$$p^2 = -\frac{m\omega\hbar}{2}\left(a^{+2} - a^{+}a - aa^{+} - a^2\right)$$

$$\langle 0|p^2|0\rangle = \frac{m\omega\hbar}{2}\langle 0|a^2 + a^{+}a + aa^{+} - a^{+2}|0\rangle$$

We know: $a|n\rangle = \sqrt{n}\,|n-1\rangle$, $a^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$

- $\langle 0|a^2|0\rangle = 0$
- $\langle 0|a^{+}a|0\rangle = 0$
- $\langle 0|aa^{+}|0\rangle = \langle 0|a|1\rangle = \langle 0|0\rangle = 1$
- $\langle 0|a^{+2}|0\rangle = \langle 0|a^{+}|1\rangle = \langle 0|2\rangle\sqrt{2} = 0$

$$\therefore \quad \langle 0|p^2|0\rangle = \frac{m\omega\hbar}{2}$$

7)

Given that $\psi_{n,n-1,m} = A_n r^{n-1} e^{-r/na_0} Y^{m}_{n-1}(\theta,\varphi)$

Probability to find electron in r and $r+dr$ is

$$P(r) = A_n r^{2(n-1)} e^{-2r/na_0} r^2$$
$$= A_n r^{2n} e^{-2r/na_0}$$

r^2 is multiplied to account for Jacobian.

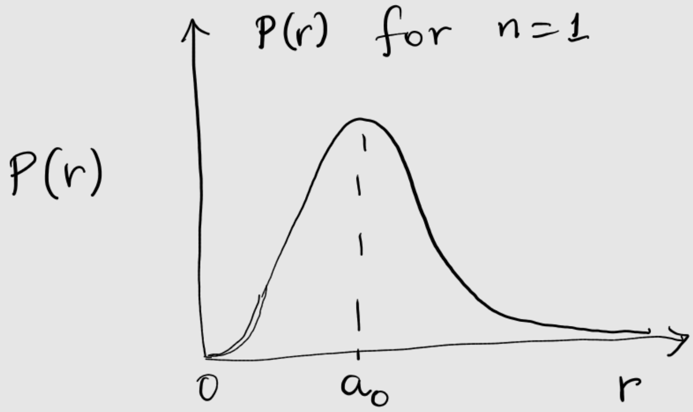
To find maximum of *P(r)*

$$\frac{d}{dr} P(r) = 0$$

$$\Rightarrow A_n \left[2n \, r^{2n-1} e^{-2r/na_0} + r^{2n} \left(-\frac{2}{na_0} \right) e^{-2r/na_0} \right] = 0$$

$$\frac{2n}{r} - \frac{2}{na_0} = 0 \quad \Rightarrow \quad r = n^2 a_0$$

$\therefore$ Probability to find electron is maximum at $n^2 a_0$



8) $\langle L_x \rangle = \langle l \, m | L_x | l \, m \rangle$

$$L_x = \frac{L_+ + L_y}{2}$$

$$\langle L_x \rangle = \frac{1}{2} \langle l \, m | L_+ + L_- | l m \rangle$$

$$L_\pm | l, m \rangle = \hbar \sqrt{(l \mp m)(l \pm m + 1)} \; | l, m \pm 1 \rangle$$

$$\langle L_x \rangle = \frac{1}{2}\hbar\sqrt{(l-m)(l+m+1)}\ \langle l,m|l,m+1\rangle$$

$$+\frac{1}{2}\hbar\sqrt{(l+m)(l-m+1)}\ \langle l,m|l,m-1\rangle$$

$$\langle l,m|l,m+1\rangle = \langle l,m|l,m-1\rangle = 0$$

- $\langle L_x \rangle = 0$

- $\langle L_x^2 \rangle = \left\langle \left(\frac{L_+ + L_-}{2}\right)^2 \right\rangle = \frac{1}{4}\langle lm| L_+^2 + L_+L_- + L_-L_+ + L_-^2 |lm\rangle$

- $\langle lm| L_+^2 |lm\rangle = \langle lm| L_-^2 |lm\rangle = 0$

$$\langle lm| L_+L_- |lm\rangle = \hbar\sqrt{(l+m)(l-m+1)}\ \langle l,m| L_+ |l,m-1\rangle$$

$$= \hbar\sqrt{(l+m)(l-m+1)}\ \hbar\sqrt{(l-m+1)(l+m)}\ \langle lm|lm\rangle$$

$$\langle L_+L_- \rangle = \hbar^2 (l+m)(l-m+1)$$

- $\langle lm| L_-L_+ |lm\rangle = \hbar\sqrt{(l-m)(l+m+1)}\ \langle lm| L_- |l,m+1\rangle$

$$= \hbar\sqrt{(l-m)(l+m+1)}\ \hbar\sqrt{(l+m+1)(l-m)}\ \langle lm|lm\rangle$$

$$\langle L_-L_+ \rangle = \hbar^2 (l-m)(l+m+1)$$

- $\langle L_x^2 \rangle = \frac{1}{4}\hbar^2\left[(l+m)(l-m+1) + (l-m)(l+m+1)\right]$

$$= \frac{\hbar^2}{4} 2\left(l(l+1) - m^2\right)$$

$$\langle L_x^2 \rangle = \frac{\hbar^2}{2}\left(l(l+1) - m^2\right)$$

$$\sigma_{L_x} = \sqrt{\langle (L_x - \langle L_x \rangle)^2 \rangle} = \sqrt{\langle L_x^2 \rangle}$$

$$\sigma_{L_x} = \frac{\hbar}{\sqrt{2}} \sqrt{l(l+1) - m^2}$$

9 a) Rotation operator for spin

$$U(\theta) = I \cos\frac{\theta}{2} - i \sin\frac{\theta}{2}\, \hat{\theta} \cdot \vec{\sigma}$$

$$\vec{\sigma} = \sigma_x \hat{i} + \sigma_y \hat{j} + \sigma_z \hat{k}$$

$\hat{\theta} = \hat{i}$ (for this problem), $\theta = \pi/2$

Then, we get

$$U\left(\frac{\pi}{2}\right) = I \cos\frac{\pi}{4} - i \sin\frac{\pi}{4}\, \sigma_x$$

$$= \frac{1}{\sqrt{2}} I - \frac{i}{\sqrt{2}} \sigma_x$$

Note $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$U\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

we need :

$$U\left(\frac{\pi}{2}\right)|\uparrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= |\uparrow\rangle - i|\downarrow\rangle$$

$$U\left(\frac{\pi}{2}\right)|\uparrow\rangle = |\uparrow\rangle - i|\downarrow\rangle$$

9b) Let $|u\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$.

$$\langle\sigma_x\rangle = \langle u|\sigma_z|u\rangle = (a^* \;\; b^*)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= (a^* \;\; -b^*)\begin{pmatrix} a \\ b \end{pmatrix} = |a|^2 - |b|^2$$

$\langle\sigma_x\rangle = 0$ implies that $|a|^2 - |b|^2 = 0$

Hence, we can choose $a = e^{i\theta_1}$, $b = e^{i\theta_2}$

$$\therefore \quad |u\rangle = \begin{pmatrix} e^{i\theta_1} \\ e^{i\theta_2} \end{pmatrix}$$

$$\langle\sigma_y\rangle = \langle u|\sigma_y|u\rangle = (e^{-i\theta_1} \;\; e^{-i\theta_2})\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} e^{i\theta_1} \\ e^{i\theta_2} \end{pmatrix}$$

$$= (ie^{-i\theta_2} \;\; -ie^{-i\theta_1})\begin{pmatrix} e^{i\theta_1} \\ e^{i\theta_2} \end{pmatrix}$$

$$= i\left[e^{i(\theta_1-\theta_2)} - e^{-i(\theta_1-\theta_2)}\right]$$

$$\langle \sigma_y \rangle = -2\sin(\theta_1-\theta_2)$$

We require that $\langle \sigma_y \rangle = 0$ or $\sin(\theta_1-\theta_2) = 0$

Now, $\langle \sigma_x \rangle = \begin{pmatrix} e^{-i\theta_1} & e^{-i\theta_2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta_1} \\ e^{i\theta_2} \end{pmatrix}$

$$\langle \sigma_x \rangle = \begin{pmatrix} e^{-i\theta_2} & e^{-i\theta_1} \end{pmatrix} \begin{pmatrix} e^{i\theta_1} \\ e^{i\theta_2} \end{pmatrix}$$

$$= 2\cos(\theta_1-\theta_2)$$

We require that $\langle \sigma_x \rangle = 0$ or $\cos(\theta_1-\theta_2) = 0$

Now, if $\theta_1 - \theta_2 = \beta$, then both conditions

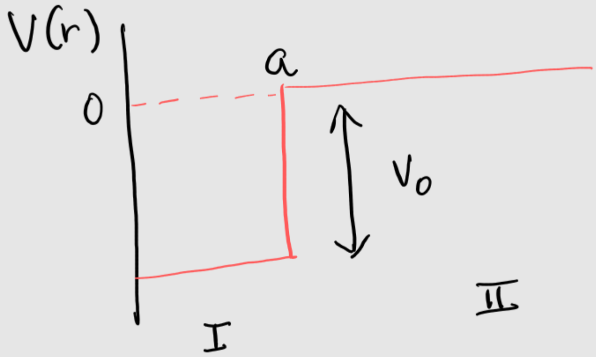
$\langle \sigma_y \rangle = \sin\beta = 0$ and

$\langle \sigma_x \rangle = \cos\beta = 0$ CANNOT be simultaneously satisfied.

Hence, it is not possible to simultaneously satisfy

$$\langle \sigma_x \rangle = \langle \sigma_y \rangle = \langle \sigma_z \rangle = 0.$$

10) Potential



- Radial Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

- We require $l=0$ case.

$$\therefore \quad -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r)\,u = Eu$$

- Applied to this problem, we have

$$-\frac{\hbar^2}{2m}\frac{d^2u_I}{dr^2} - V_0 u_I = E u_I \quad \text{for } (r \le a) \text{ region I}$$

$$-\frac{\hbar^2}{2m}\frac{d^2u_{II}}{dr^2} = E u_{II} \quad \text{for } (r > a) \text{ region II}$$

- Consider region I. Here $E < 0$. $\boxed{r \le a}$

$$\frac{d^2u_I}{dr^2} + \frac{2mV_0}{\hbar^2}u_I = -\frac{2m}{\hbar^2}(-E)\,u_I$$

$$\frac{d^2u_I}{dr^2} + \frac{2m}{\hbar^2}(V_0 - E)\,u_I = 0$$

put $\quad k_1^2 = \frac{2m}{\hbar^2}(V_0 - E)$

$$\therefore \frac{d^2 u_I}{dr^2} + k_1^2 u_I = 0 \quad \text{------ } (1)$$

- Consider region II. Here too $E < 0$. $\boxed{r > a}$

$$-\frac{\hbar^2}{2m} \frac{d^2 u_{II}}{dr^2} = (-E)\, u_{II}$$

$$\Rightarrow \frac{d^2 u_{II}}{dr^2} - \frac{2mE}{\hbar^2} u_{II} = 0 \Rightarrow \frac{d^2 u_{II}}{dr^2} - k_2^2 u_{II} = 0 \quad \text{--- } (2)$$

$$k_2^2 = 2mE/\hbar^2$$

- Solution in region I $(r \leq a)$

$$u_I(r) = A \sin k_1 r + \cancel{B \cos k_1 r}$$

Since we need $u_I(r) = 0$ at $r = 0$

- Solution in region II $(r > a)$

$$u_{II}(r) = C e^{k_2 r} + D e^{-k_2 r}$$

Match solutions at $r = a$.

$$u_I(a) = u_{II}(a)$$

$$A \sin k_1 a = D e^{-k_2 a}$$

Match derivatives at $r = a$:

$$A k_1 \cos k_1 a = -D k_2 e^{-k_2 a}$$

put $\frac{A}{D} = C$

$\therefore$ we have

$C \sin k_1 a = e^{-k_2 a}$ and $C k_1 \cos k_1 a = -k_2 e^{-k_2 a}$

Dividing the two, we get quantisation condition:

$$\boxed{\tan k_1 a = -\frac{k_1}{k_2}} \qquad \text{------} (3)$$

b) There will be no energy levels if condition in Eq (3) is violated.

i.e if $\tan k_1 a > 0$

That is, $k_1 a < \pi/2$ or $k_1^2 a^2 < \frac{\pi^2}{4}$

i.e $$\frac{2m}{\hbar^2}(V_0 - E)\, a^2 < \frac{\pi^2}{4}$$

$$\frac{2mV_0 a^2}{\hbar^2} < \frac{\pi^2}{4} + \frac{2mE}{\hbar^2} a^2$$

$$\Rightarrow \quad V_0 a^2 < \frac{\pi^2 \hbar^2}{8m}.$$