Indian Institute of Science Education and Research Pune
Mid-semester Exam, Aug (2024) semester.

Course name: Quantum Mechanics I — **Course code:** PH-3124
Date: 26.11.2024 (3:00 PM to 5:00 PM) — *Duration:* 2 hours
Instructor : M. S. Santhanam — Maximum marks: 60

- Among questions 1 to 6, answer ANY FIVE of them. Questions 7 to 10 are compulsory.
- If you draw sketches as an answer, label the axes. No marks without axes labels.
- SHOW ALL THE STEPS CLEARLY in your calculations while arriving at an answer.
- Unless specified otherwise, all the symbols have their usual meanings.
- Use the same symbols and notation given in the question. Do not use your own symbols and notations.

1. (a) Consider a rectanglular barrier potential ($V_0 > 0$)

$$\begin{aligned} V(x) &= V_0, \quad (-a \leq x \leq a) \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Sketch the potential. On the same figure, sketch the incoming plane wave ϕ (from left region) with energy $E < V_0$, wavefunction within the barrier region and the transmitted wavefunction. Sketch all the three on the same figure.
(b) In the limit of $V_0 \to \infty$, what happens to reflection and transmission coefficients R and T ? (3+2)

2. If $H(r, \theta, \phi)$ represents a Hamiltonian in three dimensions with the potential function $V(r, \theta, \phi) = r^2$, answer the following ;
(i) What are the conserved quantities ? For each conserved quantity, what are the corresponding commutator relation.
(ii) In general, for a state characterised by a value of l, what is the degree of degeneracy ? (3+2)

3. Given a wavepacket of the form $\phi(x) = A\ e^{ip_0 x/\hbar}\ e^{-|x|/(2\Delta)}$, where $0 \leq x \leq \infty$. Find $\langle x \rangle$ in terms of A. (need not calculate A). (5)

4. Show that the anti-commutator $[L_x, L_y]_+$ can be reduced to $-\frac{i}{2}\left(L_+^2 - L_-^2\right)$. (5)

5. Let

$$A = \begin{pmatrix} 1 & 3 \\ 5 & 4 \end{pmatrix}$$

Write the matrix A in terms of Pauli matrices. (5)

6. Evaluate the matrix element $\langle 0|p^2|0\rangle$, where $|0\rangle$ is the ground state of oscillator. The mass of oscillator is m and frequency is ω. (5)

7. For the hydrogen atom problem, with $l = n - 1$, the radial wavefunction has the form

$$\psi_{n,n-1,m} = A_n \; r^{n-1} \; e^{-r/na_0} \; Y^m_{n-1}(\theta, \phi).$$

In this, A_n is the normalisation constant (need not be calculated). Sketch the radial probability density for $n = 1$ state. What is the value of r at which the probability to find the electron is the maximum ? (7)

8. Assume that a particle has an orbital angular momentum with z component being $m\hbar$, and square of total angular momentum being $\hbar^2 l(l+1)$. Given that $\langle L_x \rangle = 0$, calculate $\langle L_x^2 \rangle$. Use these answers to show that the uncertainty in L_x is (8)

$$\sigma_{L_x} = \frac{\hbar}{\sqrt{2}} \sqrt{l(l+1) - m^2}.$$

9. (a) Let spin-up be denoted by $|\uparrow\rangle$ and spin-down be denoted by $|\downarrow\rangle$. Let the initial state $|\uparrow\rangle$ be rotated by angle $\pi/2$ about x-axis. Calculate the resulting state, and write it in terms of $|\uparrow\rangle$ and $|\downarrow\rangle$.
(b) Let a spin-1/2 particle be represented by a general state $\xi = \begin{pmatrix} a \\ b \end{pmatrix}$, where a and b are complex numbers. Then, show that it is impossible to satisfy that $\langle \sigma_x \rangle = \langle \sigma_y \rangle = \langle \sigma_z \rangle = 0$. (5+5)

10. A particle of mass m is placed in a finite spherical well.

$$\begin{aligned} V(r) &= -V_0, \quad \text{if } r \leq a, \\ &= 0, \qquad \text{if } r > a. \end{aligned}$$

This is a problem in three dimension, but consider only $l = 0$ case to answer the questions.
a) Solve the radial Schrodinger equation to obtain the quantisation condition from which discrete energy values can be obtained. You need not obtain the energy, only the condition is required.
(b) Show that there is no bound state if $V_0 a^2 < \pi^2 \hbar^2 / 8m$. (5+5)

Some useful information. You can use this information in your calculation (if neeeded):

Hydrogen atom wavefunction :

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r) \; Y_l^m(\theta, \phi)$$

Spherical harmonics :

$$Y_l^m(\theta, \phi) = \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l+1)!}{4\pi}} \sqrt{\frac{(l+m)!}{(2l)!(l-m)!}} e^{im\phi} (\sin\theta)^{-m} \times \frac{d^{l-m}}{d(\cos\theta)^{l-m}} (\sin\theta)^{2l}$$

Ground state of hydrogen atom : $\sqrt{1/\pi a_0^3} \; \exp(-r/a_0)$

Ladder operators acting on $|lm\rangle$: $\quad L_\pm |lm\rangle = \hbar \left[(l \mp m)(l \pm m + 1)\right]^{1/2} |l, m \pm 1\rangle$

Creation and annihilation operators : $\quad a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle.$