2D Isotropic Oscillator

$$H = \frac{p_x^2 + p_y^2}{2\mu} + \frac{1}{2}\mu\omega^2(x^2+y^2)$$

$\mu \to$ mass
$\omega \to$ frequency

- This Hamiltonian is rotationally invariant. Check if $[H, L_z]=0$.
- Due to $[H, L_z]=0$, $\hat{H}$ and $\hat{L}_z$ share common eigenstate.

In cylindrical coordinates, eigenstate can be assumed as ---- (1)

$$\psi_E(\rho,\varphi) = R(\rho)\,\Phi(\varphi)$$

Note that $L_z|\Phi(\varphi)\rangle = m\hbar\,|\Phi(\varphi)\rangle$, where $\Phi_m(\varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}}$.

- $V(x,y) = \frac{1}{2}\mu\omega^2(x^2+y^2)$.

In cylindrical-polar coordinates $V(\rho,\varphi) = \frac{1}{2}m\omega^2\rho^2 = V(\rho)$

- Schrodinger equation

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right) + V(\rho)\right]\psi_E(\rho,\varphi) = E\,\psi_E(\rho,\varphi)$$

Substitute from Eq (1), and use $\frac{\partial^2}{\partial\varphi^2}\Phi_m(\varphi) = -m^2\Phi_m(\varphi)$.

- Then, we get the RADIAL EQUATION: ---- (2)

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2}\right) + V(\rho)\right]R(\rho) = E\,R(\rho)$$

We already know the solution to angular part, i.e, $\Phi_m(\varphi)$.
We have to solve Eq (2) to find the radial eigenfunction $R(\rho)$.

- First determine the limiting cases:

As $\rho \to 0$:

Eq (2) can be rewritten as :

$$\left[\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{m^2}{\rho^2}\right)-\frac{\mu^2\omega^2}{2\hbar^2}\rho^2+\frac{2\mu}{\hbar^2}E\right]R(\rho)=0$$

As $\rho\to 0$, $\rho^2\to 0$. But $\frac{1}{\rho^2}>>1$. So $\frac{2\mu}{\hbar^2}E$ can be neglected.

$\therefore$ As $\rho\to 0$: $\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{m^2}{\rho^2}\right)R(\rho)=0$

We can verify that $R(\rho)=\rho^{\pm m}$ is a valid solution.

However ρ^{-m} diverges. So, we have $R(\rho)\xrightarrow[\rho\to 0]{}\rho^{|m|}$.

- As $\rho\to\infty$: $\left(\frac{d^2}{d\rho^2}-\frac{\mu^2\omega^2}{\hbar^2}\rho^2\right)R(\rho)=0$

Solution is $R(\rho)\xrightarrow[\rho\to\infty]{} e^{-\frac{\mu\omega}{2\hbar}\rho^2}\rho^\beta$ ($\beta\to$ constant)

$\therefore$ We assume that $R(\rho)=\rho^{|m|}e^{-\frac{\mu\omega}{2\hbar}\rho^2}U(\rho)$

- Now, make ρ dimensionless with the change of variables

$$y=\sqrt{\frac{\mu\omega}{\hbar}}\,\rho \quad \text{and} \quad \varepsilon=\frac{E}{\hbar\omega}$$

Radial equation (2) for $U(y)$ becomes

$$\frac{d^2U}{dy^2}+\left[\left(\frac{2|m|+1}{y}\right)-2y\right]\frac{dU}{dy}+(2\varepsilon-2|m|-2)U=0$$

- Use power series method to solve this eqn:

$$U(y)=\sum_{r=0}^{\infty}c_r y^r$$

- Obtain the recurrence relation:

$$C_{r+2} = \frac{2r - (2\varepsilon - 2|m| - 2)}{(r+2)\left[2|m| + r + 2\right]} C_r$$

- This series must terminate at finite r if $y \to \infty$ behaviour should come out right.

$$2\varepsilon - 2|m| - 2 = 2r$$

$$\therefore \Rightarrow \varepsilon = r + |m| + 1$$

NOTE: r must be an even number, i.e, $r = 2k$

$$\therefore \frac{E}{\hbar\omega} = (2k + |m| + 1) \Rightarrow E_k = (2k + |m| + 1)\hbar\omega$$

$k = 0, 1, 2, ----$

↓ radial quantum number

- put $n = 2k + |m|$. $E_n = (n+1)\hbar\omega$

- For a given n, what are allowed values of |m|
  " " ", degeneracy is $(n+1)$.

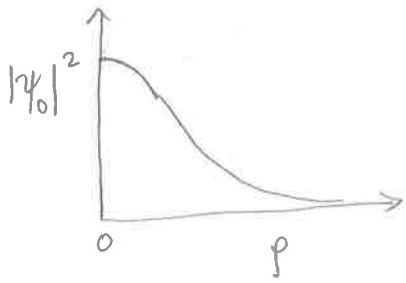
k	m	n	E_n
0	0	0	$1\hbar\omega$
0	1	1	$2\hbar\omega$
0	-1	1	$2\hbar\omega$
0	2	2	$3\hbar\omega$
0	-2	2	$3\hbar\omega$
1	0	2	$3\hbar\omega$

- Exercise: Show in general that for a given value of n, degree of degeneracy is $n+1$.

- Eigenfunctions:

- $\boxed{n=0}$

$$\psi_0(\rho,\varphi) = \frac{C_0}{\sqrt{2\pi}}\, e^{-\frac{\mu\omega}{2\hbar}\rho^2}$$



Normalise to get C_0:

$$C_0 = \sqrt{2\mu\omega/\hbar}$$

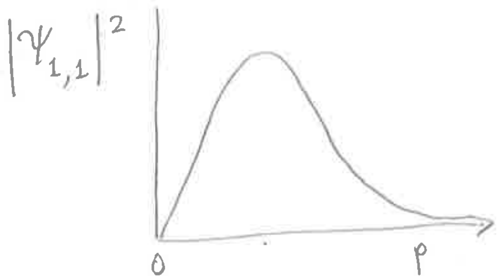
- Agrees with cartesian coordinate result.

- $\boxed{n=1}$ has $k=0$ and $m=\pm 1$

$$\psi_{n=1,\,m=1}(r,\varphi) = \frac{C_0}{\sqrt{2\pi}}\sqrt{\frac{\mu\omega}{\hbar}}\,\rho\, e^{-\frac{\mu\omega}{2\hbar}\rho^2} e^{i\varphi}$$

$$\psi_{n=1,\,m=-1}(r,\varphi) = \;\; '' \qquad '' \qquad '' \;\; e^{-i\varphi}$$

$$C_0 = \sqrt{2\mu\omega/\hbar}$$



- In cartesian coordinates:

$$\left.\begin{aligned}\psi_{10} &= \frac{\sqrt{2}\,\mu\omega}{\hbar\sqrt{\pi}}\, e^{-\frac{\mu\omega}{2\hbar}\rho^2}\rho\cos\varphi \\ \psi_{01} &= \;\; '' \;\; e^{-\frac{\mu\omega}{2\hbar}\rho^2}\rho\sin\varphi\end{aligned}\right\}$$

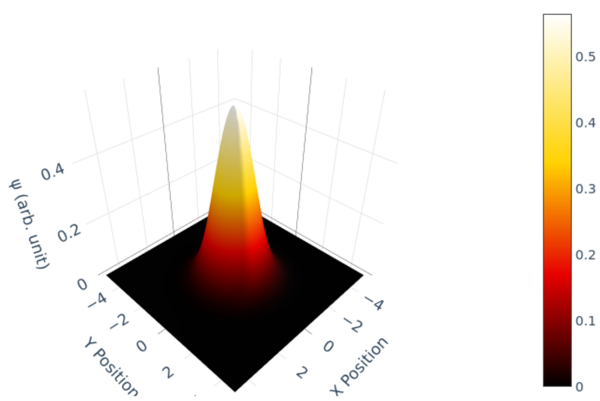
use $\rho^2 = x^2+y^2$
$x = \rho\cos\varphi$
$y = \rho\sin\varphi$

$\therefore$

$$\psi_{n=1,\,m=1} = \frac{1}{\sqrt{2}}\left(\psi_{10} + i\psi_{01}\right)$$

$$\psi_{n=1,\,m=-1} = \frac{1}{\sqrt{2}}\left(\psi_{10} - i\psi_{01}\right)$$

2D Wave Function for $(n_x, n_y) = (0, 0)$ in a Harmonic Oscillator



Note that in the ground state probability changes in radial direction, but is a constant along angular direction, that along any circle.