- Momentum operator

position operator	$\hat{x}$	$i\hbar \frac{\partial}{\partial p}$
Momentum operator	$\underbrace{-i\hbar \frac{\partial}{\partial x}}_{\text{in position representation}}$	$\underbrace{\hat{p}}_{\text{in momentum representation}}$

- How is momentum operator $-i\hbar \frac{\partial}{\partial p}$?

<u>Consider this to be a definition:</u>

Here is a dynamical variable : u

Corresponding operator : $\hat{u}$

Mean value or expectation value of $\hat{u}$: $\langle u \rangle$

$$\langle u \rangle = \int \psi^*(u) \; \hat{u} \; \psi(u) \, du$$

Here, $\psi(u) = \langle u | \psi \rangle$ is some state *u* representation

- Applying this to momentum operator

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(p) \; \hat{p} \; \psi(p) \, dp \qquad (1)$$

write $\psi(p)$ as Fourier transform of momentum eigenstate $\varphi(x)$:

$$\psi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \; \varphi(x) \; e^{-ipx/\hbar}$$

Then, $$\psi^*(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \; \varphi^*(x) \; e^{ipx/\hbar}$$

Using Eq. (1):

$$\langle p \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \underbrace{\int_{-\infty}^{\infty} dx' \; \varphi^*(x') \; e^{-ipx'/\hbar}}_{\psi^*(p)} \; \hat{p} \underbrace{\int_{-\infty}^{\infty} dx \; \varphi(x) \; e^{-ipx/\hbar}}_{\psi(p)}$$

By definition: $\hat{u} \; |u\rangle = u |u\rangle$ (2)

$|u\rangle \rightarrow$ state

$u \rightarrow$ scalar number

Then, what is $\hat{u} \; \psi(u)$?

$$\hat{u} \; |u\rangle = u \; |u\rangle$$

$$\hat{u} \; \psi(u) = \langle u | \hat{u} | \psi \rangle = \int \underbrace{\langle u | \hat{u} | u' \rangle}_{\text{By Eq. 2}} \langle u' | \psi \rangle \, du'$$

By Eq. 2

$$\hat{u} \; |u'\rangle = u' \; |u'\rangle$$

$$\therefore \; \hat{u} \; \psi(u) = \int u' \; \langle u | u' \rangle \; \langle u' | \psi \rangle \; du'$$

$$= \int u' \; \delta(u-u') \; \langle u' | \psi \rangle \; du'$$

$$= u \; \langle u | \psi \rangle$$

$$\therefore \; \hat{u} \; \psi(u) = u \; \psi(u) \qquad (3)$$

If $\hat{u} = \hat{p}$, then $\hat{p} \; \psi(p) = p \; \psi(p)$

Using Eq. (3) $\quad \hat{p}\left(\frac{e^{-ipx/\hbar}}{\sqrt{2\pi}}\right) = p\left(\frac{e^{-ipx/\hbar}}{\sqrt{2\pi}}\right)$

$$\langle p \rangle = \frac{1}{2\pi}\int dp\, p \int dx'\, \varphi^*(x')\, e^{-ipx'/\hbar} \underbrace{\int dx\, \varphi(x)\, e^{-ipx/\hbar}}_{\text{Integrate this by parts}}$$

Let $\quad u = \varphi(x) \qquad dv = e^{-ipx/\hbar}\, dx$

Then,

$$\int dx\, \varphi(x)\, e^{-ipx/\hbar} = \underbrace{\cancel{\varphi(x)\,\frac{e^{-ipx/\hbar}}{(ip/\hbar)}\Bigg|_{-\infty}^{\infty}}}_{=0 \text{ since } \varphi(x)=0 \text{ at } x=\infty,\ x=-\infty} + \int \frac{e^{-ipx/\hbar}}{(ip/\hbar)}\, \frac{d\varphi}{dx}\, dx$$

$$\langle p \rangle = \frac{1}{2\pi}\int dp \int dx'\, \varphi^*(x')\, e^{-ipx'/\hbar} \left(\frac{\hbar}{i}\right) \int e^{ipx/\hbar}\, \frac{d\varphi}{dx}\, dx$$

Perform *p* integral using the fact that

$$\frac{1}{2\pi}\int e^{-ip(x-x')/\hbar}\, dp = \delta(x'-x)$$

$$\langle p \rangle = \iint dx'\, dx\, \varphi^*(x') \left(\frac{\hbar}{i}\right) \frac{d\varphi}{dx}\, \delta(x'-x)$$

$$= \int dx\, \varphi^*(x) \left(\frac{\hbar}{i}\frac{d}{dx}\right) \varphi(x)$$

Comparing with Equation (1), we recognise that

$\hat{p} \longrightarrow -i\hbar \dfrac{d}{dx}$ is the momentum operator in position representation