Sample Questions: Quantum Mechanics I

Q1) Show that the area element $dx\,dy$ transforms as $\rho\,d\rho\,d\phi$ for the coordinate transformation $x = \rho\cos(\phi)$, $y = \rho\sin(\phi)$.

Solution:

Given the transformations:

$$x = \rho\cos(\phi), \quad y = \rho\sin(\phi),$$

we want to show that the area element $dx\,dy$ in Cartesian coordinates transforms to $\rho\,d\rho\,d\phi$ in polar coordinates. To find how the area element transforms, we need to compute the Jacobian's determinant of the transformation from (x, y) to (ρ, ϕ). The Jacobian determinant J is defined as:

$$J = \det\begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} \end{pmatrix}.$$

Calculate the partial derivatives:

$$\frac{\partial x}{\partial \rho} = \cos(\phi), \quad \frac{\partial x}{\partial \phi} = -\rho\sin(\phi),$$
$$\frac{\partial y}{\partial \rho} = \sin(\phi), \quad \frac{\partial y}{\partial \phi} = \rho\cos(\phi).$$

Substitute these into the Jacobian matrix:

$$J = \det\begin{pmatrix} \cos(\phi) & -\rho\sin(\phi) \\ \sin(\phi) & \rho\cos(\phi) \end{pmatrix}.$$

Evaluate the determinant:

$$\begin{aligned} J &= \cos(\phi)\cdot(\rho\cos(\phi)) - (-\rho\sin(\phi))\cdot\sin(\phi) \\ &= \rho\cos^2(\phi) + \rho\sin^2(\phi). \end{aligned}$$

Using the trigonometric identity $\cos^2(\phi) + \sin^2(\phi) = 1$, we have:

$$J = \rho.$$

The area element $dx\,dy$ transforms as:

$$dx\,dy = |J|\,d\rho\,d\phi.$$

Since $J = \rho$, we get:

$$dx\,dy = \rho\,d\rho\,d\phi.$$

We have shown that the area element $dx\,dy$ in Cartesian coordinates transforms to $\rho\,d\rho\,d\phi$ in polar coordinates.

Try: To derive the volume element dV in spherical coordinates (r, θ, ϕ) from Cartesian coordinates (x, y, z), we start with the transformations:

$$x = r\sin(\theta)\cos(\phi), \quad y = r\sin(\theta)\sin(\phi), \quad z = r\cos(\theta).$$

The volume element in Cartesian coordinates is given by $dV = dx\,dy\,dz$. To find how this transforms into spherical coordinates, we need to compute the Jacobian determinant of the transformation from (r, θ, ϕ) to (x, y, z).

The Jacobian matrix J of the transformation is defined as:

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix}.$$

compute the partial derivatives:

$$\begin{aligned} \frac{\partial x}{\partial r} &= \sin(\theta)\cos(\phi), & \frac{\partial x}{\partial \theta} &= r\cos(\theta)\cos(\phi), & \frac{\partial x}{\partial \phi} &= -r\sin(\theta)\sin(\phi), \\ \frac{\partial y}{\partial r} &= \sin(\theta)\sin(\phi), & \frac{\partial y}{\partial \theta} &= r\cos(\theta)\sin(\phi), & \frac{\partial y}{\partial \phi} &= r\sin(\theta)\cos(\phi), \\ \frac{\partial z}{\partial r} &= \cos(\theta), & \frac{\partial z}{\partial \theta} &= -r\sin(\theta), & \frac{\partial z}{\partial \phi} &= 0. \end{aligned}$$

The Jacobian matrix J becomes:

$$J = \begin{pmatrix} \sin(\theta)\cos(\phi) & r\cos(\theta)\cos(\phi) & -r\sin(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) & r\cos(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) \\ \cos(\theta) & -r\sin(\theta) & 0 \end{pmatrix}.$$

The determinant of the Jacobian can be calculated as:

$$|J| = r^2\sin(\theta).$$

Thus, the volume element in spherical coordinates is given by:

$$dV = |J|\,dr\,d\theta\,d\phi = r^2\sin(\theta)\,dr\,d\theta\,d\phi.$$

Q2) Show that $\delta(cx) = \frac{1}{|c|}\delta(x)$, $y\delta'(y) = -\delta(y)$, and $y\delta(y) = 0$

Solution:

Part 1: Show that $\delta(cx) = \frac{1}{|c|}\delta(x)$

The Dirac delta function $\delta(x)$ satisfies the property that for any function $f(x)$:

$$\int_{-\infty}^{\infty} \delta(x) f(x)\,dx = f(0).$$

we start by making a substitution $u = cx$, which implies:

$$du = c\,dx \quad \Rightarrow \quad dx = \frac{du}{c}.$$

Substitute this into the integral:

$$\int_{-\infty}^{\infty} \delta(cx) f(x)\, dx = \int_{-\infty}^{\infty} \delta(u) f\left(\frac{u}{c}\right) \frac{du}{|c|}.$$

Using the property of the delta function $\delta(u)$, we have:

$$\int_{-\infty}^{\infty} \delta(u) f\left(\frac{u}{c}\right)\, du = f(0).$$

Therefore, the integral becomes:

$$\int_{-\infty}^{\infty} \delta(cx) f(x)\, dx = \frac{1}{|c|} f(0).$$

Now consider the integral involving $\delta(x)$ directly:

$$\int_{-\infty}^{\infty} \frac{1}{|c|} \delta(x) f(x)\, dx = \frac{1}{|c|} f(0).$$

Since both integrals yield the same result, we conclude:

$$\delta(cx) = \frac{1}{|c|} \delta(x).$$

Part 2: Show that $y\delta'(y) = -\delta(y)$

We start by considering the integral:

$$\int_{-\infty}^{\infty} y\delta'(y) f(y)\, dy.$$

Using integration by parts, let:

$$u = y \quad \Rightarrow \quad du = dy,$$
$$dv = \delta'(y) f(y)\, dy \quad \Rightarrow \quad v = f(y).$$

Applying the integration by parts formula $\int u\, dv = uv - \int v\, du$, we obtain:

$$\int_{-\infty}^{\infty} y\delta'(y) f(y)\, dy = [y f(y) \delta(y)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(y) f(y)\, dy.$$

We need to evaluate the boundary term:

$$[y f(y) \delta(y)]_{-\infty}^{\infty}.$$

The Dirac delta function $\delta(y)$ is non-zero only at $y = 0$, so the product $y\delta(y)$ is zero everywhere, including at $y = 0$, since multiplying y by $\delta(y)$ results in zero. Thus, the boundary term is:

$$[y f(y) \delta(y)]_{-\infty}^{\infty} = 0.$$

Substituting the result of the boundary term back into the integration by parts formula, we get:

$$\int_{-\infty}^{\infty} y\delta'(y) f(y)\, dy = -\int_{-\infty}^{\infty} \delta(y) f(y)\, dy.$$

The sifting property of the Dirac delta function states that:

$$\int_{-\infty}^{\infty} \delta(y) f(y)\, dy = f(0).$$

Therefore, we have:

$$\int_{-\infty}^{\infty} y\delta'(y) f(y)\, dy = -f(0).$$

Since this holds true for any test function $f(y)$, this implies:

$$y\delta'(y) = -\delta(y).$$

Part 3: Show that $y\delta(y) = 0$

The delta function $\delta(y)$ is nonzero only at $y = 0$. Therefore, multiplying y (which is zero at $y = 0$) by $\delta(y)$:

$$\int_{-\infty}^{\infty} y\delta(y) f(y)\, dy = 0,$$

for any test function $f(y)$. Thus, we have:

$$y\delta(y) = 0.$$

Q3) Show that:

$$[L_i, P_j] = i\hbar\epsilon_{ijk} P_k,$$

where ϵ_{ijk} is the Levi-Civita symbol, defined as:

$$\epsilon_{ijk} = \begin{cases} 1, & \text{for even permutations of } (123), \\ -1, & \text{for odd permutations of } (123), \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

The angular momentum operator $\mathbf{L}$ is given by:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

where $\mathbf{r}$ is the position operator and $\mathbf{p}$ is the linear momentum operator.

$$\begin{aligned} L_1 &= r_2 p_3 - r_3 p_2 \\ &= \epsilon_{123} r_2 p_3 + \epsilon_{132} r_3 p_2 \end{aligned}$$

$$\begin{aligned} L_2 &= r_3 p_1 - r_1 p_3 \\ &= \epsilon_{213} r_1 p_3 + \epsilon_{231} r_3 p_1 \end{aligned}$$

$$\begin{aligned} L_3 &= r_1 p_2 - r_2 p_1. \\ &= \epsilon_{312} r_1 p_2 + \epsilon_{321} r_2 p_1 \end{aligned}$$

The components of $\mathbf{L}$ are given by:

$$L_i = \epsilon_{ijk} r_j p_k.$$

We need to find the commutation relation $[L_i, p_j]$. By substituting the expression for L_i, we have:

$$[L_i, p_j] = [\epsilon_{ilm} r_l p_m, p_j].$$

$$[L_i, p_j] = \epsilon_{ilm} [r_l p_m, p_j].$$

Utilizing the product rule for commutators $[AB, C] = A[B, C] + [A, C]B$, we get:

$$[L_i, P_j] = \epsilon_{ilm} \left(r_l [p_m, p_j] + [r_l, p_j] p_m \right).$$

Substituting these results back into the expression for the commutator $[L_i, P_j]$:

$$\begin{aligned} [L_i, P_j] &= \epsilon_{ilm} \left(r_l \cdot 0 + (i\hbar \delta_{lj}) p_m \right) \\ &= i\hbar \epsilon_{ilm} \delta_{lj} p_m. \end{aligned}$$

Now, summing over l using the property of the Kronecker delta:

$$[L_i, P_j] = i\hbar \epsilon_{ijm} p_m.$$

Finally, we can rearrange this to match the desired form:

$$[L_i, P_j] = i\hbar \epsilon_{ijk} P_k,$$

where we relabeled the index m as k.

Q4) Show that $[L_x, L_y] = i\hbar L_z$ and in general $[L_i, L_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} L_k$

Solution: The angular momentum operators in quantum mechanics are defined as:

$$L_x = Y P_z - Z P_y,$$

$$L_y = Z P_x - X P_z,$$

$$L_z = X P_y - Y P_x,$$

where X, Y, and Z are position operators, and P_x, P_y, and P_z are the corresponding momentum operators.

Calculate $[L_x, L_y]$

We start with the expression for the commutator:

$$[L_x, L_y] = [Y P_z - Z P_y, Z P_x - X P_z].$$

Expanding this using the linearity of the commutator:

$$[L_x, L_y] = \underbrace{[Y P_z, Z P_x]}_{1} - \underbrace{[Y P_z, X P_z]}_{2} - \underbrace{[Z P_y, Z P_x]}_{3} + \underbrace{[Z P_y, X P_z]}_{4}.$$

1st Term: $[Y P_z, Z P_x]$

We start with the commutator:

$$[YP_z, ZP_x] = Y[P_z, ZP_x] + [Y, ZP_x]P_z.$$

Now we need to calculate the two commutators involved: $[P_z, ZP_x]$ and $[Y, ZP_x]$.

$$[P_z, ZP_x] = [P_z, Z]P_x + Z[P_z, P_x].$$

$$[P_z, ZP_x] = (-i\hbar)P_x + Z(0) = -i\hbar P_x.$$

Again, using the product rule:

$$[Y, ZP_x] = [Y, Z]P_x + Z[Y, P_x].$$

$$[Y, ZP_x] = 0 \cdot P_x + Z(0) = 0.$$

Now we can substitute back into our expression for $[YP_z, ZP_x]$:

$$[YP_z, ZP_x] = Y(-i\hbar P_x) + 0 \cdot P_z = -i\hbar YP_x.$$

2nd Term: $[YP_z, XP_z]$

$$[YP_z, XP_z] = Y[P_z, XP_z] + [Y, XP_z]P_z = 0.$$

3rd Term: $[ZP_y, ZP_x]$

$$[ZP_y, ZP_x] = Z[P_y, ZP_x] + [Z, ZP_x]P_y = 0.$$

4th Term: $[ZP_y, XP_z]$

$$[ZP_y, XP_z] = Z[P_y, XP_z] + [Z, XP_z]P_y = i\hbar XP_y.$$

This simplifies to:

$$[L_x, L_y] = -i\hbar YP_x + i\hbar XP_y = i\hbar L_Z.$$

Q5) Show that $G^\dagger = G$ using the translation operator to order ϵ. The translation operator to first order in ϵ is given by: $T(\epsilon) = I - \frac{i\epsilon G}{\hbar}$.

Solution: The adjoint (Hermitian conjugate) of the translation operator is:

$$T^\dagger(\epsilon) = \left(I - \frac{i\epsilon G}{\hbar}\right)^\dagger = I + \frac{i\epsilon G^\dagger}{\hbar}.$$

To show that $G^\dagger = G$, we utilize the property of the translation operators, which states that the product of the translation operator and its adjoint should yield the identity operator:

$$T^\dagger(\epsilon)T(\epsilon) = I.$$

Substituting our expressions for $T(\epsilon)$ and $T^\dagger(\epsilon)$:

$$\begin{aligned} T^\dagger(\epsilon)T(\epsilon) &= \left(I + \frac{i\epsilon G^\dagger}{\hbar}\right)\left(I - \frac{i\epsilon G}{\hbar}\right) \\ &= I - \frac{i\epsilon G}{\hbar} + \frac{i\epsilon G^\dagger}{\hbar} - \frac{\epsilon^2 G^\dagger G}{\hbar^2} \\ &\approx I + \left(-\frac{i\epsilon G}{\hbar} + \frac{i\epsilon G^\dagger}{\hbar}\right), \quad \text{(neglecting higher order terms).} \end{aligned}$$

$$T^\dagger(\epsilon)T(\epsilon) \approx I + \frac{i\epsilon(G^\dagger - G)}{\hbar} = I.$$

For the above equality to hold, the term $\frac{i\epsilon(G^\dagger - G)}{\hbar}$ must vanish:

$$G^\dagger - G = 0 \implies G^\dagger = G,$$

which indicates that G is a Hermitian operator.

Q6) Prove that if $[\mathcal{P}, H] = 0$, a system that starts out in an even/odd state of parity maintains its parity under time evolution.

Solution: The operator $\mathcal{P}$ reflects the spatial coordinate x to $-x$. Mathematically, this is expressed as:

$$\mathcal{P}\psi(x) = \psi(-x),$$

where $\psi(x)$ represents the wave function of the quantum state in the position basis. The action of the parity operator $\mathcal{P}$ on a wavefunction $\psi(x)$ can be summarized as follows:

$$\mathcal{P}\psi(x) = \begin{cases} \psi(x), & \text{if } \psi(x) \text{ is even (i.e., } \psi(x) = \psi(-x)); \\ -\psi(x), & \text{if } \psi(x) \text{ is odd (i.e., } \psi(x) = -\psi(-x)). \end{cases}$$

The time evolution of a quantum state $\psi(t)$ is given by:

$$\psi(t) = \begin{cases} e^{-iHt/\hbar}\psi(0) & \text{for time-independent } H, \\ T\left\{e^{-i\int_0^t H(t')dt'/\hbar}\right\}\psi(0) & \text{for time-dependent } H. \end{cases}$$

Given that the initial state $\psi(0)$ has a definite parity, either even or odd. Lets discuss in detail both the cases.

Case 1: Even Parity

If $\psi(0)$ is an even function, it satisfies:

$$\mathcal{P}\psi(0) = \psi(0).$$

$$\mathcal{P}\psi(t) = \mathcal{P}\left(e^{-iHt/\hbar}\psi(0)\right)$$

we start with the fact that the Hamiltonian H and the parity operator $\mathcal{P}$ commute:

$$[\mathcal{P}, H] = \mathcal{P}H - H\mathcal{P} = 0.$$

This commutation relation implies that $\mathcal{P}$ and H can be interchanged in an operator expression. Now, consider the time evolution operator $e^{-iHt/\hbar}$, which is defined by its Taylor series expansion:

$$e^{-iHt/\hbar} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{iHt}{\hbar}\right)^n.$$

Applying the parity operator $\mathcal{P}$ to $e^{-iHt/\hbar}$:

$$\mathcal{P}\left(e^{-iHt/\hbar}\right) = \mathcal{P}\left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{iHt}{\hbar}\right)^n\right).$$

Since $\mathcal{P}$ and H commute, we can move $\mathcal{P}$ inside each term of the series expansion:

$$\mathcal{P}\left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{iHt}{\hbar}\right)^n\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{iHt}{\hbar}\right)^n \mathcal{P} = e^{-iHt/\hbar}\mathcal{P}.$$

This shows that:

$$\mathcal{P}e^{-iHt/\hbar} = e^{-iHt/\hbar}\mathcal{P}.$$

Hence, when we apply $\mathcal{P}$ to the time-evolved state $\psi(t)$, we have:

$$\begin{aligned}
\mathcal{P}\psi(t) &= \mathcal{P}\left(e^{-iHt/\hbar}\psi(0)\right) \\
&= e^{-iHt/\hbar}\mathcal{P}\psi(0) \\
&= e^{-iHt/\hbar}\psi(0) \\
&= \psi(t).
\end{aligned}$$

This shows that if the initial state has even parity, the state at time t also has even parity:

$$\mathcal{P}\psi(t) = \psi(t).$$

Case 2: Odd Parity

If $\psi(0)$ is an odd function, it satisfies:

$$\mathcal{P}\psi(0) = -\psi(0).$$

Again, applying the time evolution operator $e^{-iHt/\hbar}$ to the state and using the commutation relation $[\mathcal{P}, H] = 0$, we have:

$$\begin{aligned}
\mathcal{P}\psi(t) &= \mathcal{P}\left(e^{-iHt/\hbar}\psi(0)\right) \\
&= e^{-iHt/\hbar}\mathcal{P}\psi(0) \quad (\text{since } [\mathcal{P}, H] = 0) \\
&= e^{-iHt/\hbar}\left(-\psi(0)\right) \\
&= -e^{-iHt/\hbar}\psi(0) \\
&= -\psi(t).
\end{aligned}$$

This demonstrates that if the initial state has odd parity, the state at time t also has odd parity:

$$\mathcal{P}\psi(t) = -\psi(t).$$

Q7) Show that if $\mathcal{P}$ and H commute, then $\psi(-x)$ is also a solution with the same eigenvalue as $\psi(x)$.

Solution:
The time-independent Schrödinger equation is given by:

$$H\psi(x) = E\psi(x),$$

where E is the energy eigenvalue associated with the eigenstate $\psi(x)$.
Since $\mathcal{P}$ and H commute, we have:

$$\mathcal{P}H\psi(x) = H\mathcal{P}\psi(x).$$

The parity operator acts on a wave function as follows:

$$\mathcal{P}\psi(x) = \psi(-x).$$

Now applying H to both sides of the equation:

$$H(\mathcal{P}\psi(x)) = H\psi(-x)$$

Using the commutation relation, we can rewrite this as:

$$\mathcal{P}H\psi(x) = H\psi(-x).$$

Substituting $H\psi(x) = E\psi(x)$:

$$H\psi(-x) = \mathcal{P}(E\psi(x)) = E\mathcal{P}\psi(x) = E\psi(-x).$$

This shows that $\psi(-x)$ also satisfies the time-independent Schrödinger equation with the same eigenvalue E:

$$H\psi(-x) = E\psi(-x).$$
