Lecture 2.

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Contents

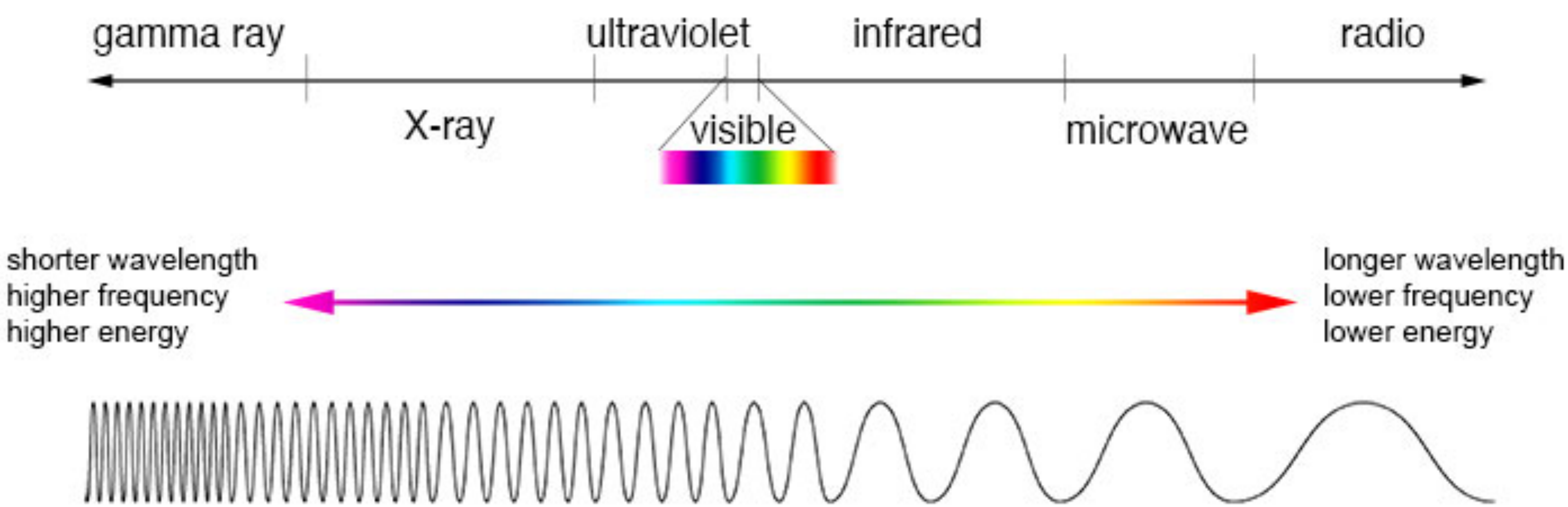
1. Maxwell's equations in Vacuum.
2. Electro-magnetic wave equation.
3. Poynting vector.
4. Irradiance.

Literature:

1. Optics, (Eugene Hecht and A. R. Ganesan)
2. Optical Physics, (A. Lipson, S. G. Lipson and H. Lipson)

ElectroMagnetic fields (EMF): Maxwell's equations (MEs)

- Light is an electro magnetic wave.
- Electro-magnetic spectrum



- Maxwell's theory of light propagation
- We should get a wave equation out of Maxwell's equations.

ElectroMagnetic fields (EMF): Maxwell's equations (MEs)

Recap of electricity and magnetism.

- Force exerted by the electric field **E** on a point charge *q is*

$$\mathbf{F}_E = q\mathbf{E}$$

- Force exerted by the magnetic field **B** on a point charge *q* moving with a velocity *v* (Lorentz force)

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}$$

- Electric fields can be created by both electric charges and **time-varying magnetic fields.**

- Magnetic fields can be created by both electric currents and **time-varying electric fields.**

ElectroMagnetic fields (EMF): Maxwell's equations (MEs)

Differential forms of MEs in vacuum (No charges and no currents)

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

Gauss's law

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}$$ Faraday's law of induction

$$\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial E}{\partial t}$$ Ampere's law

- These two equations we need to solve to get the time dependent fields.
- The time rate of change of electric field is associated with a current called *the displacement current, results in a magnetic field.*
- Apart from magnitudes, the role of **E** and **B** is identical. In other words the equations are symmetric with respect to **E** and **B**.

ElectroMagnetic fields (EMF): Maxwell's equations (MEs)

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$$\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial E}{\partial t}$$

Ampere's law

- The electric and magnetic fields mutually perpendicular to each other and also perpendicular to the direction of propagation. For a plane wave, it can be easily shown that,

$$\nabla \times \mathbf{E} = i\mathbf{k} \times \mathbf{E}$$

Transverse nature of **E** and **B** in an EM-disturbance.

ElectroMagnetic wave equation

$$\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial E}{\partial t}$$

One can also generalise this in the presence of sources.

$$\nabla \times \nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial (\nabla \times E)}{\partial t}$$

$$\boxed{\nabla \times (\nabla \times \ldots) = \nabla(\nabla \cdot) - \nabla^2}$$

$$\underbrace{\nabla(\nabla \cdot \mathbf{B})}_{\text{zero}} - \nabla^2 \mathbf{B} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

$$c^2 = \frac{1}{\mu_0 \epsilon_0}$$

$$\boxed{\nabla^2 \mathbf{B} = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2}}$$

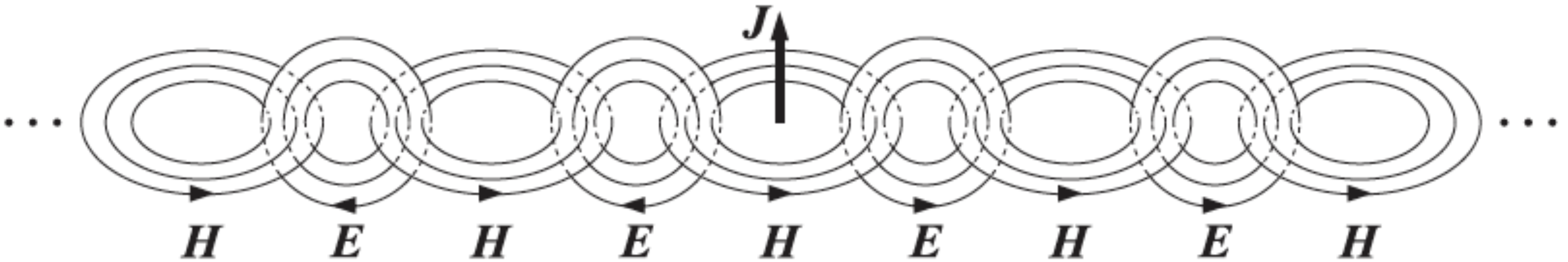
Similarly one obtains for electric field.

ElectroMagnetic wave equation

$$\nabla^2 \mathbf{B} = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Time varying current $\boldsymbol{j}$(x,t)



The cross linked B and E propagate away from the source

ElectroMagnetic wave: Potentials

Vector and Scalar potentials provide an alternative way to describe an Electro magnetic wave.

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V$$

the potentials are not uniquely defined

In vacuum V=0, hence the vector potential defines the EM-field.

Gauge invariance

$$V' = V - \frac{\partial \chi}{\partial t}$$

$$\mathbf{A}' = \mathbf{A} + \nabla \chi$$

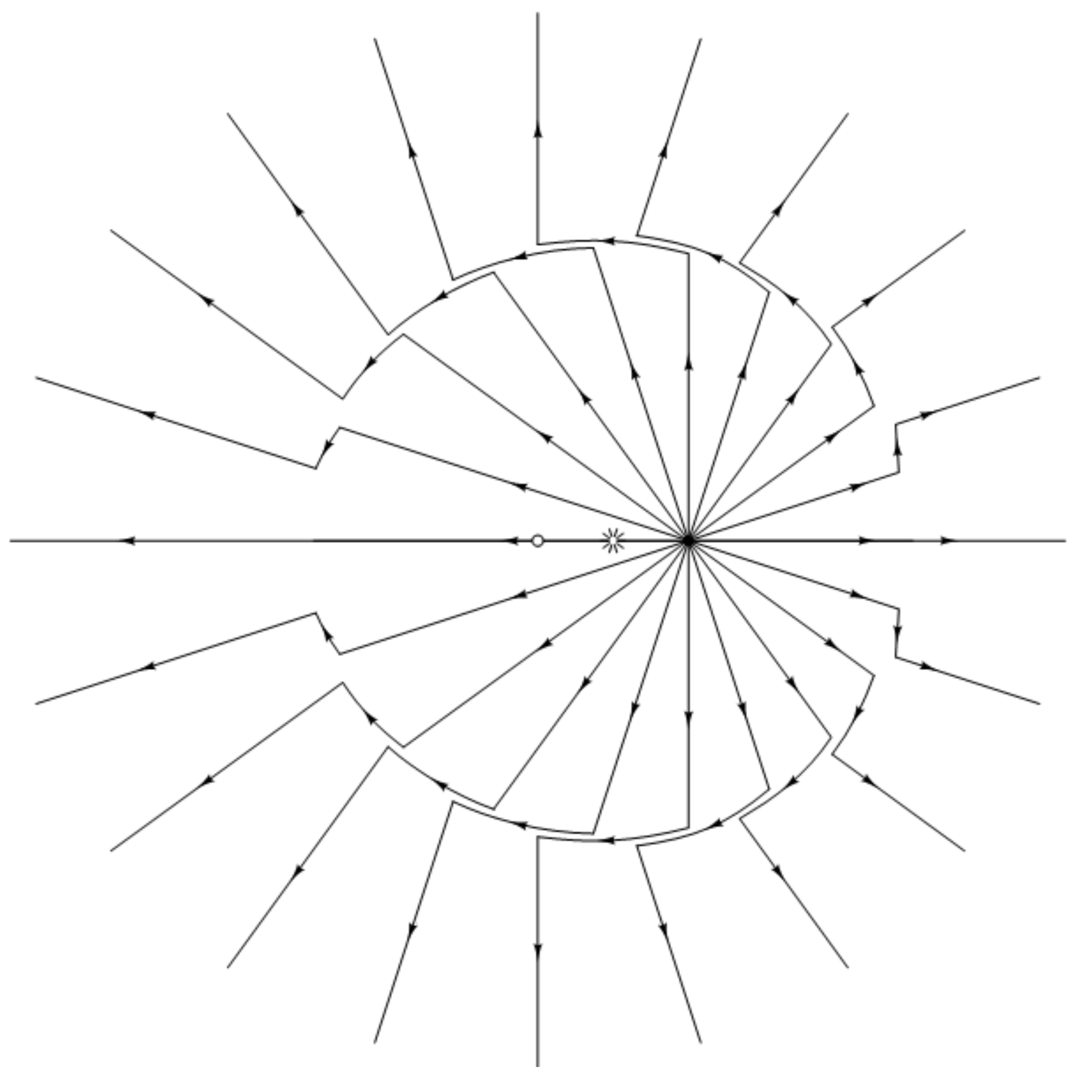
Gauge transformation

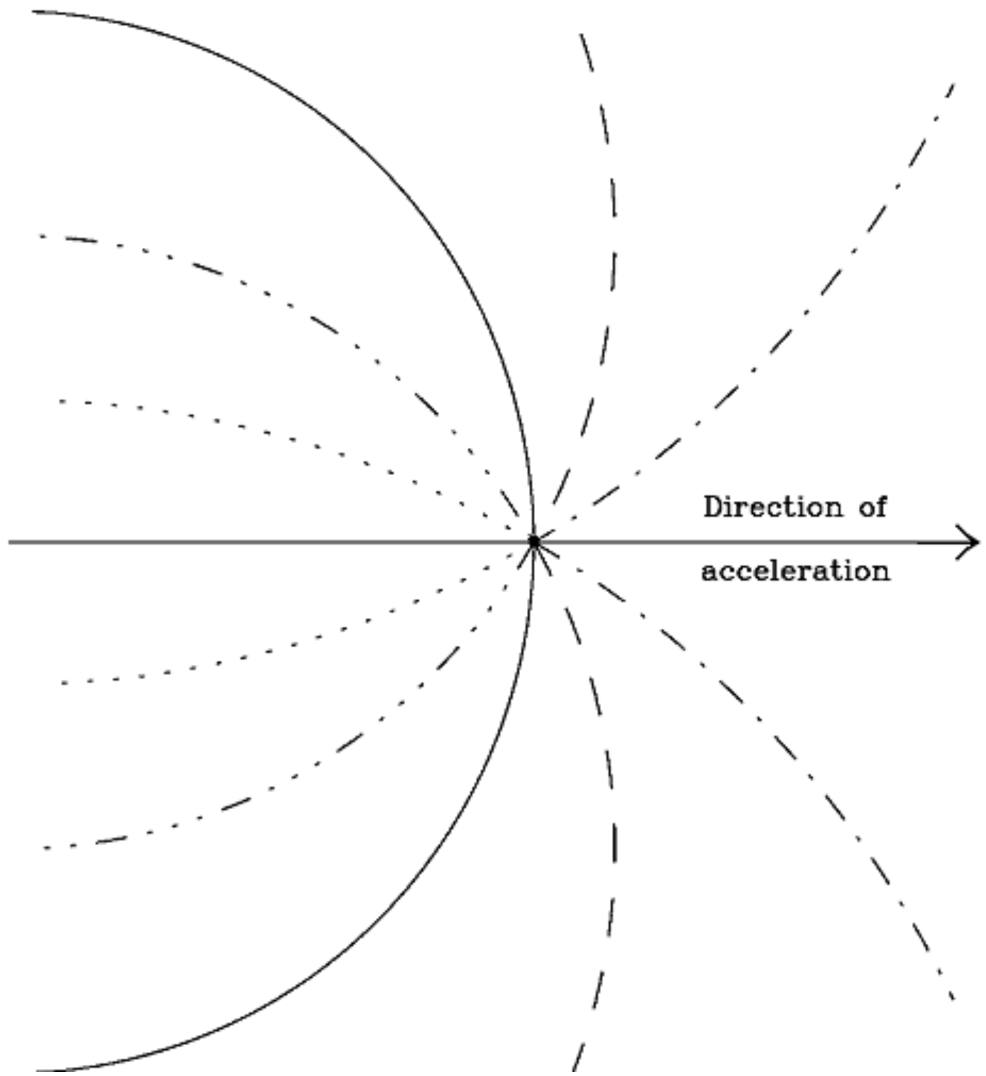
Potentials also satisfy a wave equation

The physical results do not depend on the Gauge.

ElectroMagnetic wave:

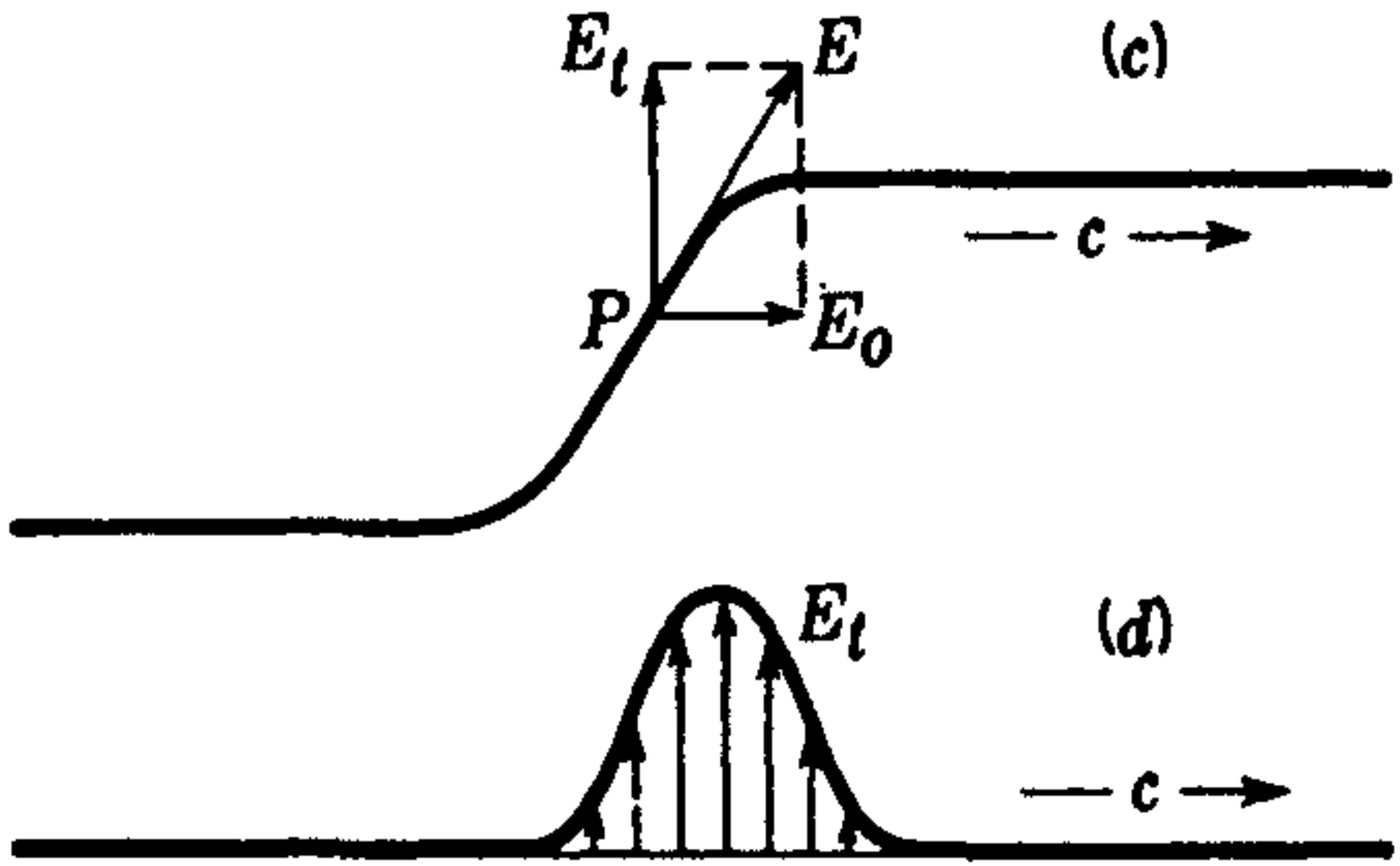
EM radiation is a disturbances (kinks) in field-lines





ElectroMagnetic wave:

EM radiation is a disturbances (kinks) in field-lines



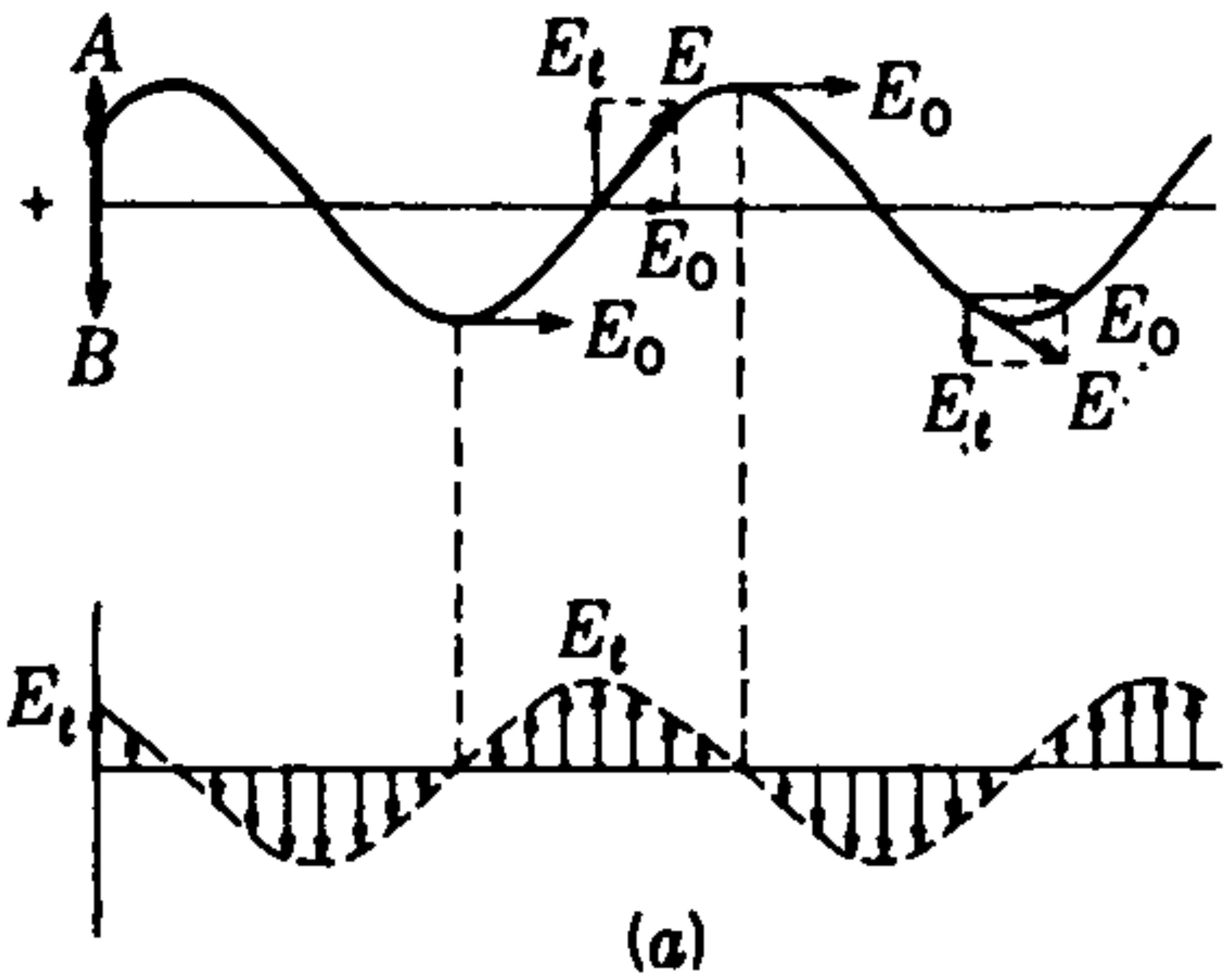
Fundamentals of Optics, Jenkins and white

Across the kink we create a pulse of EM radiation.

The amplitude of the pulse depends on how fast you accelerate the charge.

ElectroMagnetic wave:

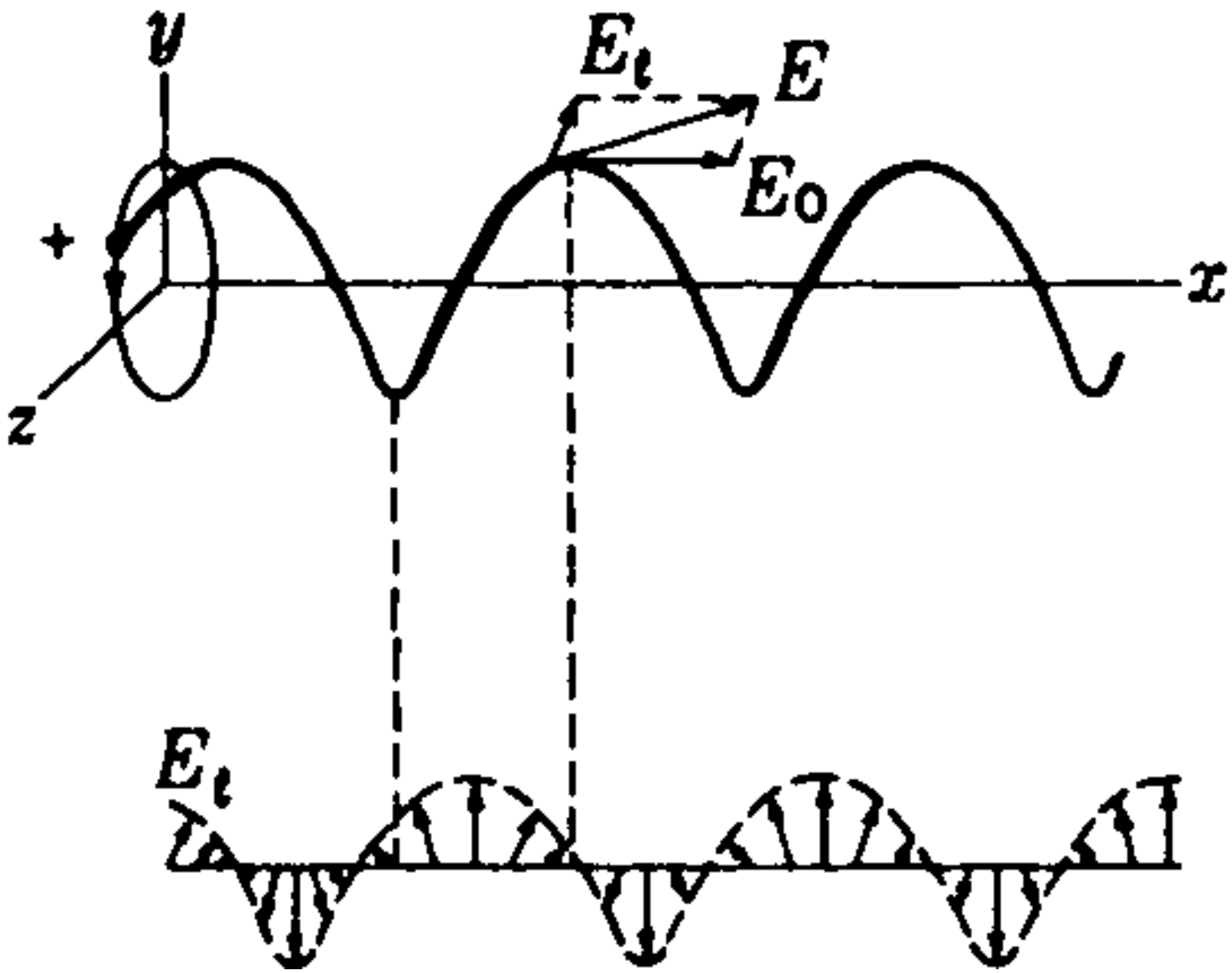
EM radiation is a disturbances (kinks) in field-lines



Plane Polarized wave by periodic motion of a single charge.

ElectroMagnetic wave:

EM radiation is a disturbances (kinks) in field-lines



Circularly Polarized wave by periodic motion of a single charge.

ElectroMagnetic wave: Energy density and Poynting vector

- An EM wave carries both energy and momentum.
- They can carry this energy forward to millions of millions of miles.
- Poynting vector: represents the directional energy flux density

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$$

- Direction of ***S*** gives the direction of energy flux.
- Magnitude of ***S*** gives the amount of energy passed through unit area per unit time. The unit of ***S*** is *Watt*/*m*^{2}.

  - The effect of a given energy-density depends on which radiation you are talking about. *For instance, 1 Watt/m^2 of 550 nm (green light) might warm our skin, but the same of 300 nm UV radiation will severely burn you in few minutes.* ***The wavelength is an important factor!!!***

ElectroMagnetic wave: Energy density and Poynting vector

- Let's take the example of harmonic plane wave:

$$\mathbf{E} = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

$$\mathbf{B} = \mathbf{B}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

The detectors measure the integrated energy flux over some time.

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E}_0 \times \mathbf{B}_0 \cos^2(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

determines the instantaneous flow of energy density

When some thing is oscillating, we talk about average values:

(useful when talking about long-time behaviours)

$$\langle \cos^2(\mathbf{k} \cdot \mathbf{r} - \omega t) \rangle = 1/2$$

$$\langle \mathbf{S} \rangle = \frac{1}{2\mu_0} \mathbf{E}_0 \times \mathbf{B}_0$$

The magnitude of ⟨S⟩ is called *irradiance (I)*.

ElectroMagnetic wave: Irradiance and units

$$I = |\langle \mathbf{S} \rangle| = \frac{1}{2\mu_0} |\mathbf{E}_0 \times \mathbf{B}_0| \quad = \frac{c\epsilon_0}{2} E_0^2$$

The unit of I is *Watt*/m^2.

- Photosynthesis or chemical reactions in which light is supplying as the energy for a reaction, the unit of "*Einstein*" is used.

> *1 Einstein* is equal to
> the energy of one mole of photons.

- Most of the cases "Watt/m^2" is not of much use unless you specify the wavelength.
- Then we need "Watt/m^2/nm", typical wave lengths are order of nm.
- Some times we need "photons/m^2/nm", for that one need to convert *Watt* $\Rightarrow$ *no. of photons*. How??

ElectroMagnetic wave: Irradiance and units

- $Watt \Rightarrow no.\ of\ photons$. How??

divide by hc/λ. But you need to know the wavelength!!

Photosynthetically active radiation (PAR) detector which detects a wavelength range from *400 nm* to *700 nm*. PAR is used by photosynthetic organisms in the process of photosynthesis.

Spectra would look completely different in different units!!!

Which is more fundamental??? wavelength or frequency???

Lets consider light moving through a medium of refractive index n

$$\lambda\nu = c/n$$

Speed of light changes in the medium, what should change on the left side??

ElectroMagnetic wave: Irradiance and units

$$\lambda\nu = c/n$$

Speed of light changes in the medium, what should change on the left side??

It is the *wavelength* which found to be changing, and *frequency* is considered more fundamental to light.

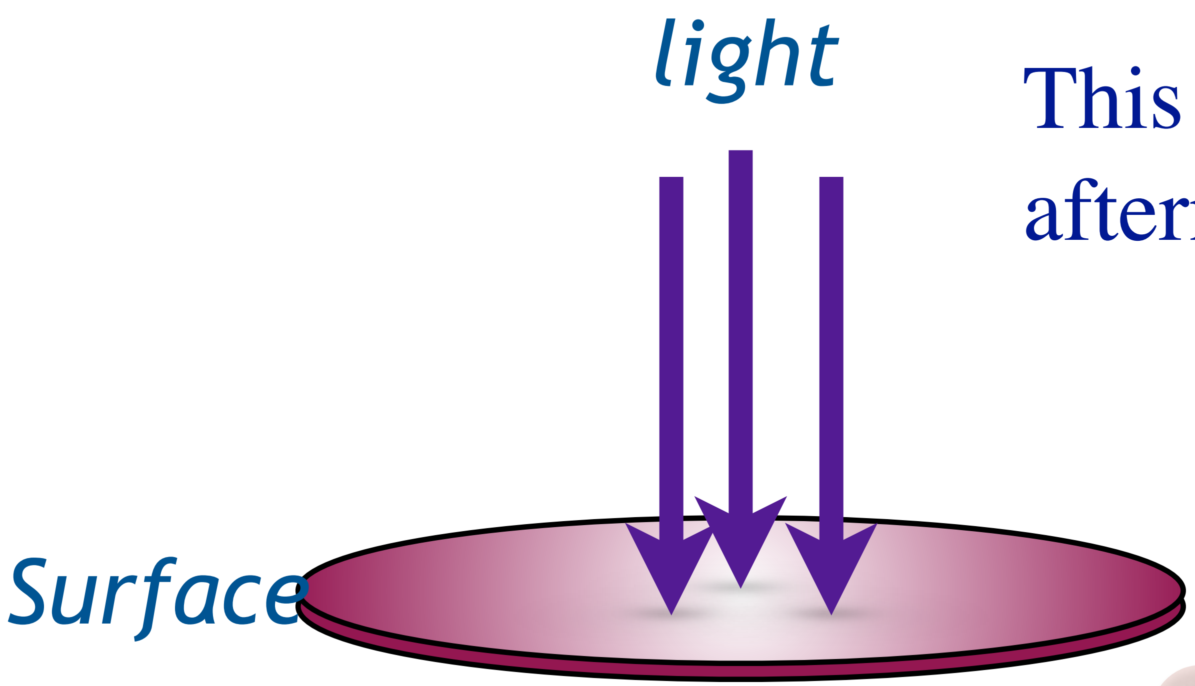
For instance, a 550 nm "green" light in vacuum has a wavelength of 414 nm in water.

The unit of irradiance is "*Watt/m²*" or "*Photons/s/cm²*".

- In front of a fire, the warmth of your skin is proportional to irradiance.
- The brightness of a white paper is proportional to the irradiance at its surface.

ElectroMagnetic wave: Irradiance and units

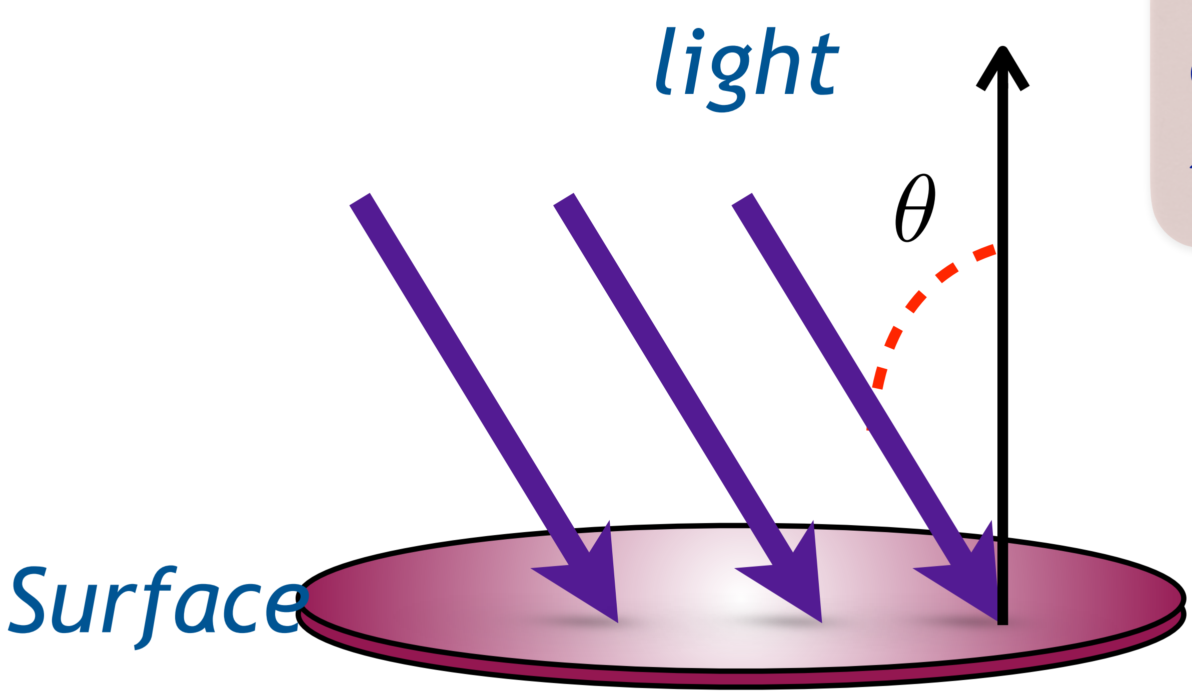
Vector irradiance



This explains why you feel hotter in the afternoon than in the morning or evening.

Let the irradiance be I_0.

Vector irradiance depends on the orientation of the surface relative to the illumination.

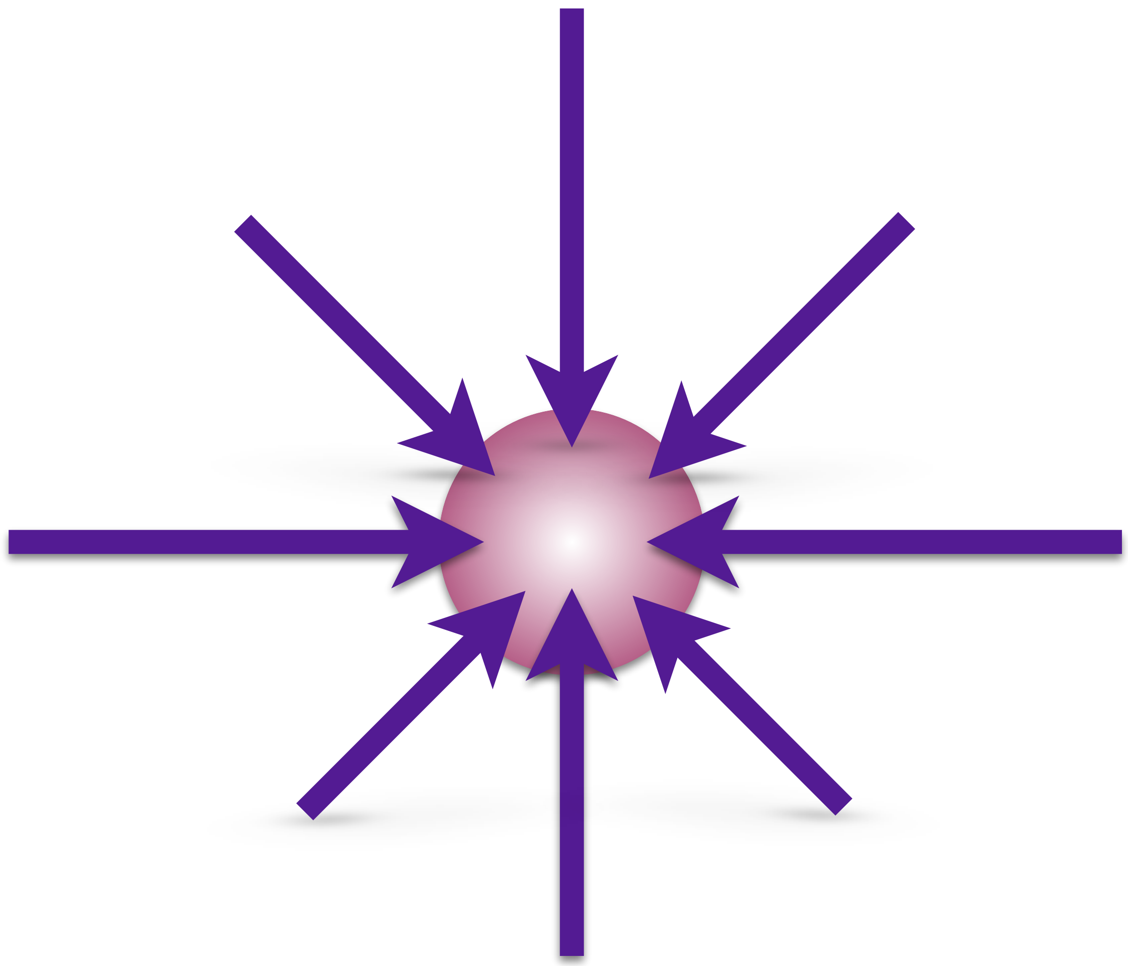


What is it for this case?

$$I = I_0 \cos\theta$$

ElectroMagnetic wave: Irradiance and units

Scalar irradiance (amount of light passing through a spherical body)



direction independent

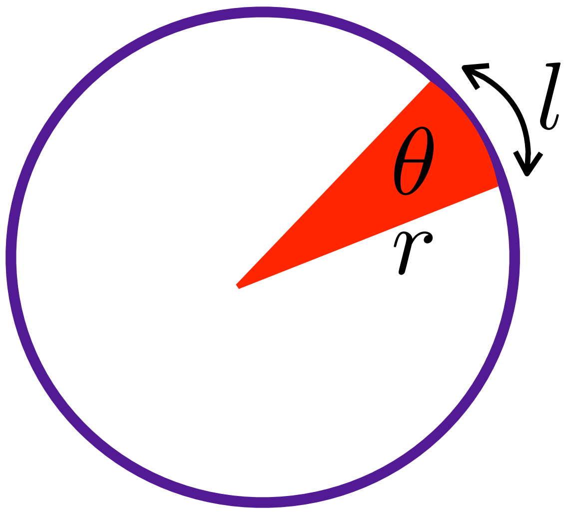
Inverse square law

The irradiance from a point source is proportional to $1/r^2$.

Verify yourself using the spherical wave solution.

ElectroMagnetic wave: Radiance

Radians and steradians



$$\theta = l/r$$

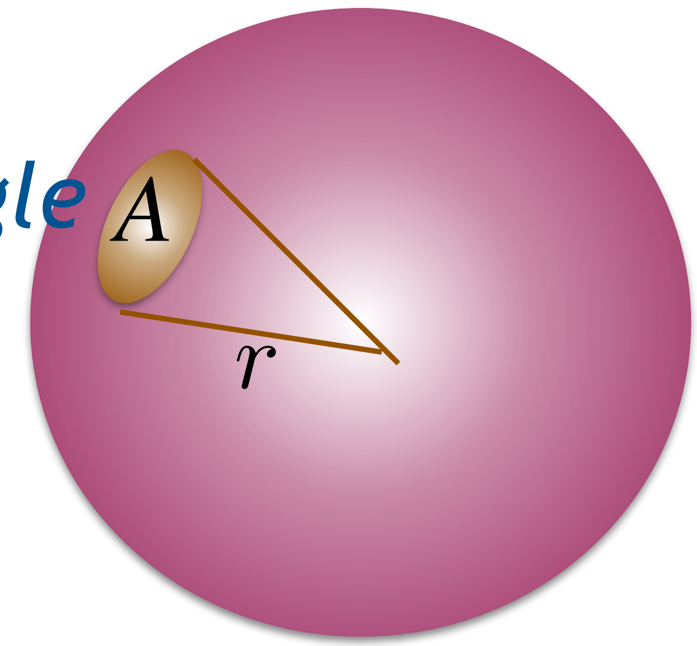
radians

2D analog of angle: Solid-angle

$$\Omega = A/r^2$$

steradians

sphere



Radiance is the energy density per solid angle normal to the beam.

Polarization

Linear Polarization

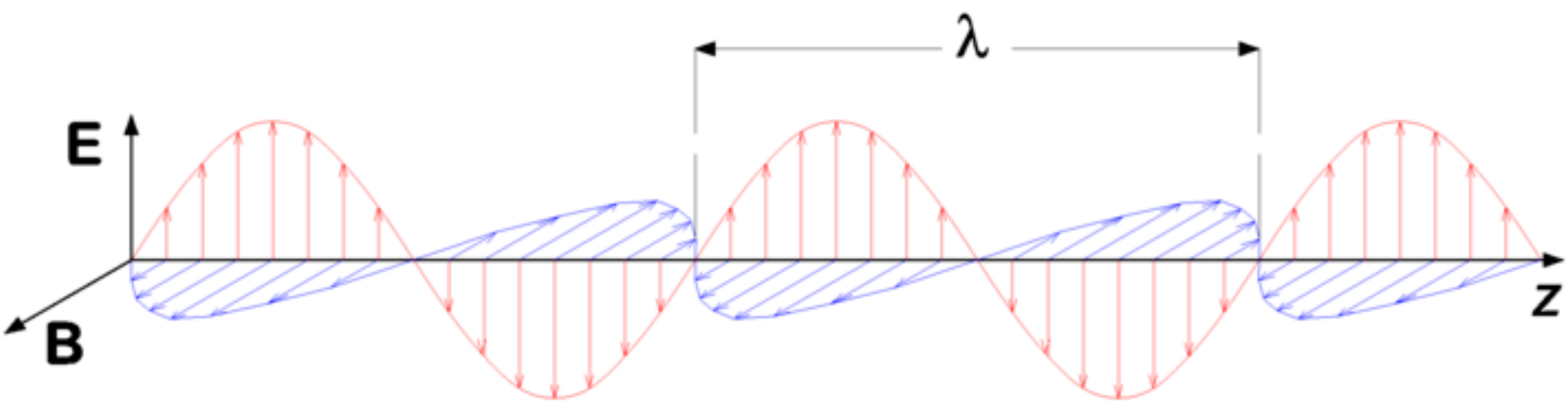
- Let's take the example of harmonic plane wave:

$$\mathbf{E} = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

$$\mathbf{B} = \mathbf{B}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

If the amplitudes $\mathbf{E}_0$ and $\mathbf{B}_0$ are constant real vectors, the wave is called **linearly polarised or plane polarised**.

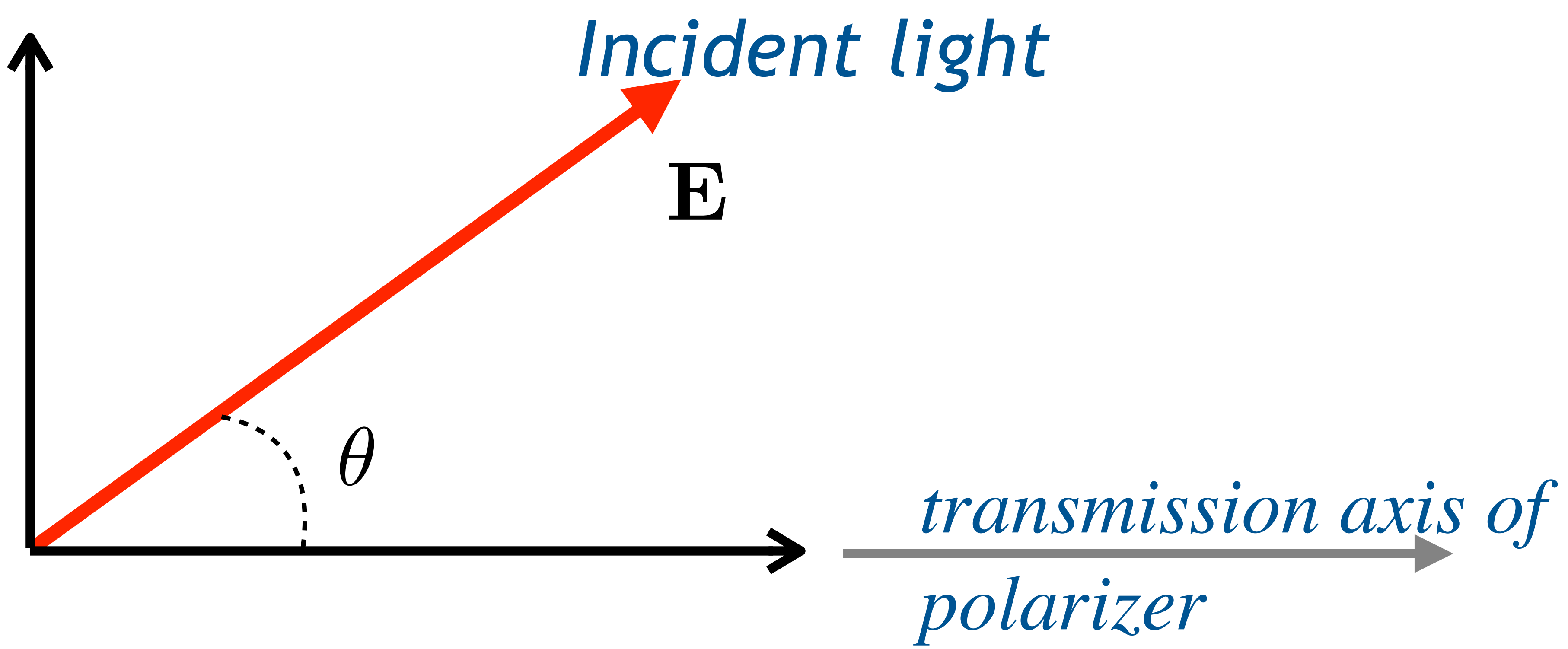
- It is now customary to define the direction of electric field as the direction of polarization.



Linear Polarization

What are unpolarised light?

What is a polariser?



E cosθ is transmitted and E sinθ is absorbed.

(Malu's law experiment)

This mechanism of anisotropic optical absorption is called ***dichroism***.

Circular polarisation

Consider two linearly polarised lights of same amplitude:

$$\mathbf{E}_1 = E_0 \cos(kz - \omega t)\hat{x}$$

$$\mathbf{E}_2 = E_0 \sin(kz - \omega t)\hat{y}$$

the resultant field is (vector addition)

$$\mathbf{E} = E_0 \left[\cos(kz - \omega t)\hat{x} + \sin(kz - \omega t)\hat{y}\right]$$

is also a solution of the wave equation.

What happen to the magnitude and direction of the resulting field at a point in space???

The above field can be seen as single one with a constant magnitude at a point in space, but the direction is rotating with a frequency ω. This type of wave is called circularly polarised.

Circular polarisation

$$\mathbf{E} = E_0\left[\cos(kz - \omega t)\hat{x} + \sin(kz - \omega t)\hat{y}\right]$$

- Is this clockwise or anti-clockwise rotation in time for a fixed space point?
- How does it behave at a given instant of time, but as a function of *z*?

This wave is called *right circularly polarised*. (right handed spirals)

- What is then left circularly polarised light?

$$\mathbf{E} = E_0\left[\cos(kz - \omega t)\hat{x} - \sin(kz - \omega t)\hat{y}\right]$$

Elliptic polarisation

It is the most general case.

$$\mathbf{E}_1 = E_0 \cos(kz - \omega t)\hat{x}$$

$$\mathbf{E}_2 = E_0' \sin(kz - \omega t)\hat{y}$$

Amplitudes are different!

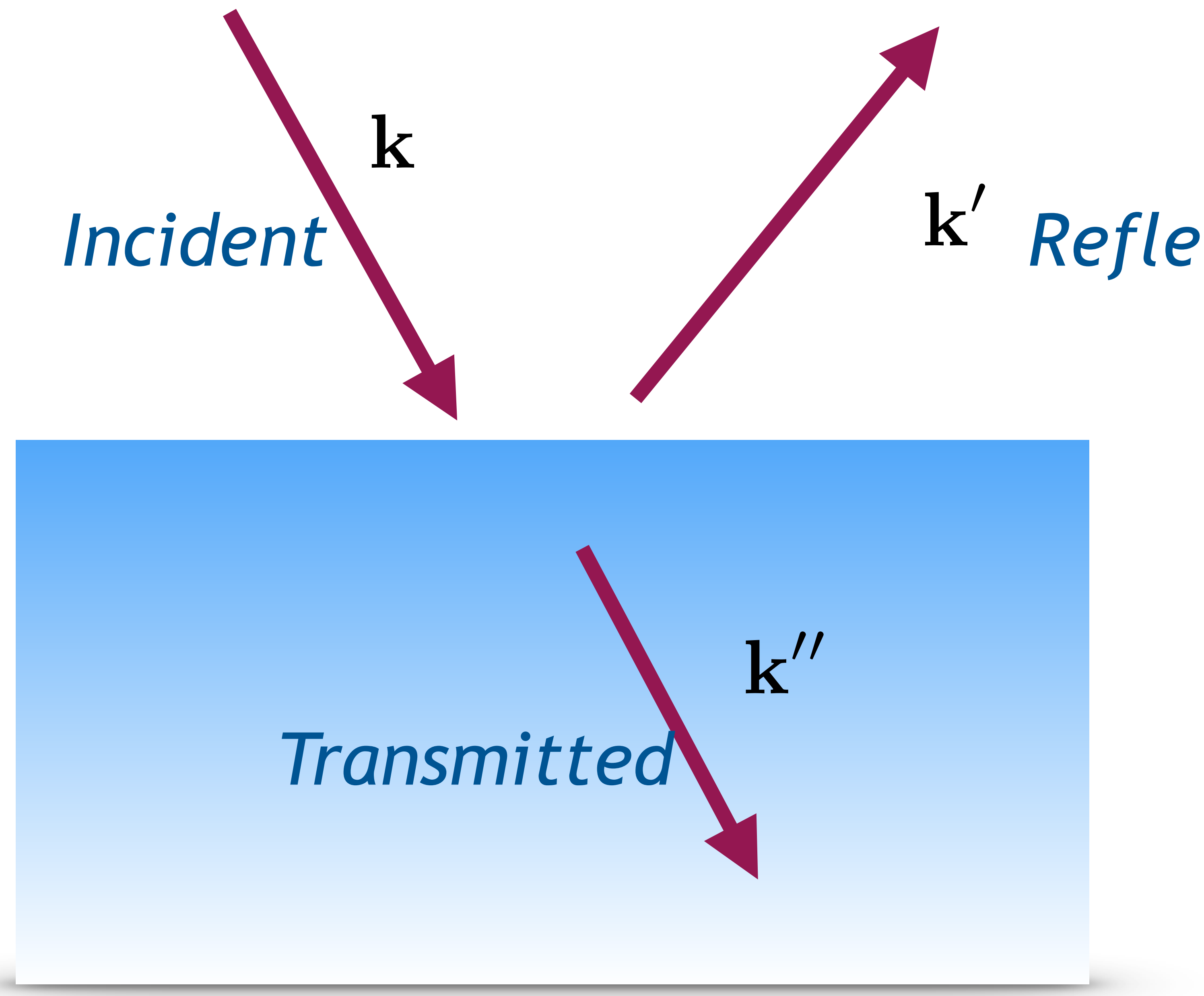
$E_0 = E_0'$ circularly polarised light.

- How does it behave at *z=0* as a function of time when $E_0 > E_0'$??

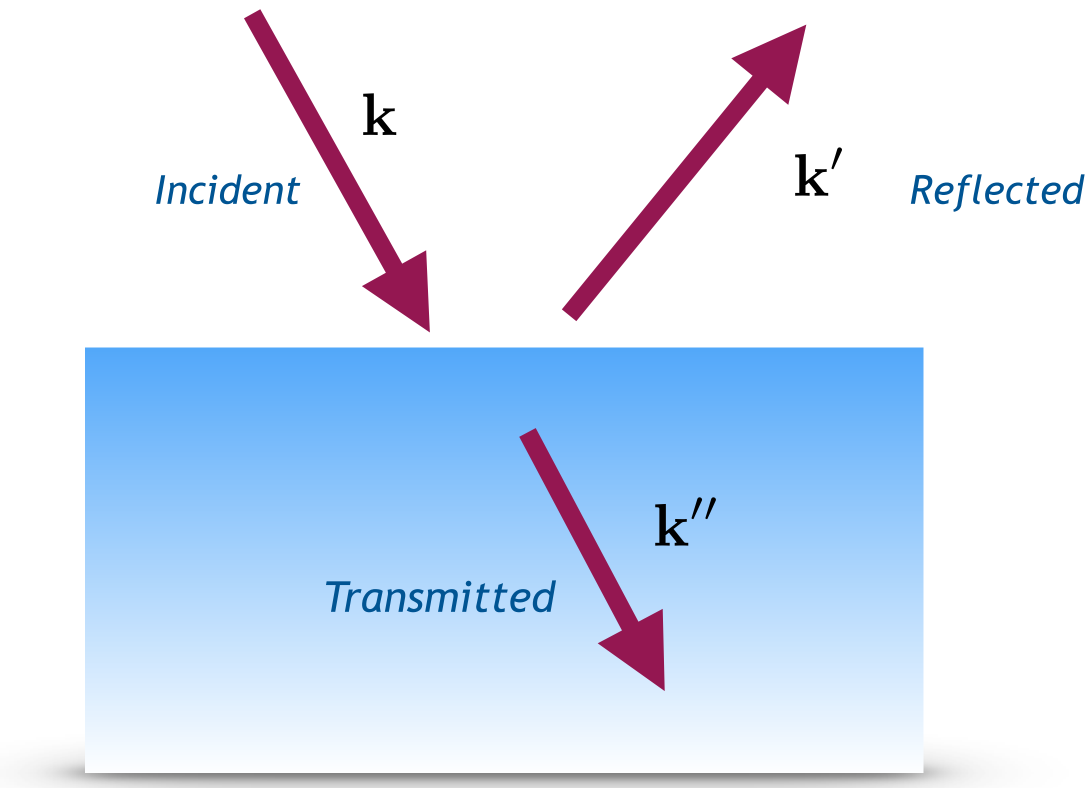
Light at the interface

Basic processes:
Transmission
Reflection
Refraction

All related to one process called the scattering of light!
(Rayleigh scattering)



Light at the interface



$$\exp\left[i(\mathbf{k}\cdot\mathbf{r}-\omega t)\right]$$ *Incident*

$$\exp\left[i(\mathbf{k}'\cdot\mathbf{r}-\omega t)\right]$$ *Reflected*

$$\exp\left[i(\mathbf{k}''\cdot\mathbf{r}-\omega t)\right]$$ *Transmitted*

- We are looking for a relation between $\mathbf{k}$, $\mathbf{k}'$, and $\mathbf{k}''$.
- The only way it can happen is at the boundary in the form:

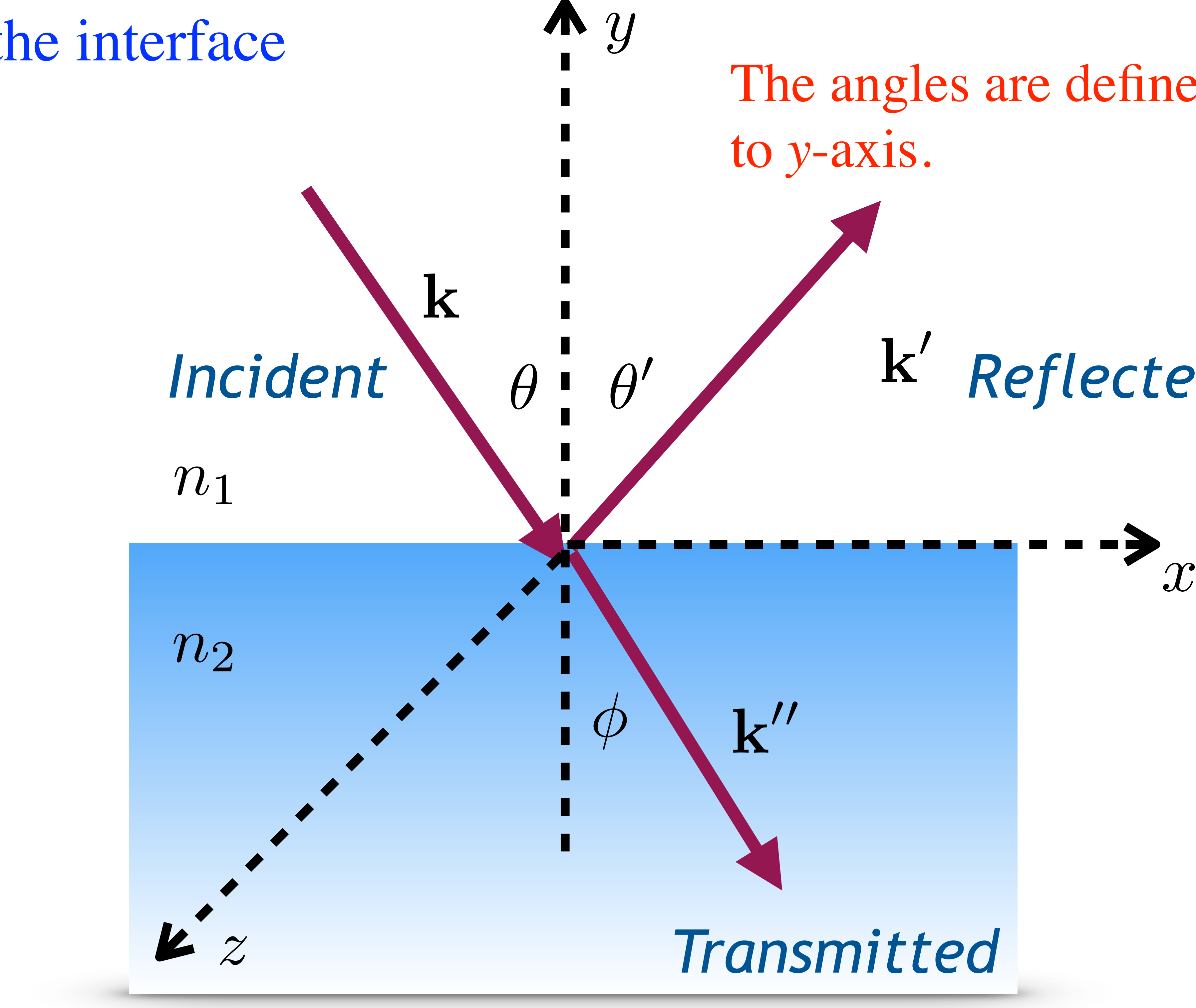
$$\mathbf{k}\cdot\mathbf{r}=\mathbf{k}'\cdot\mathbf{r}=\mathbf{k}''\cdot\mathbf{r}$$

(at the points of incidence at any time)

$$|\mathbf{k}|=|\mathbf{k}'|$$

since the two waves are in the same medium.

Light at the interface



The angles are defined with respect to y-axis.

$$|\mathbf{k}| = k$$

$$|\mathbf{k}'| = k'$$

$$|\mathbf{k}''| = k''$$

The previous condition demands that

$$k \sin\theta = k' \sin\theta' = k'' \sin\phi$$

Light at the interface

$$k \sin\theta = k' \sin\theta' = k'' \sin\phi$$

Law of Reflection

$$k \sin\theta = k' \sin\theta' \quad \Longrightarrow \quad \sin\theta = \sin\theta' \Longrightarrow \quad \theta = \theta'$$

Angle of incidence = angle of reflection

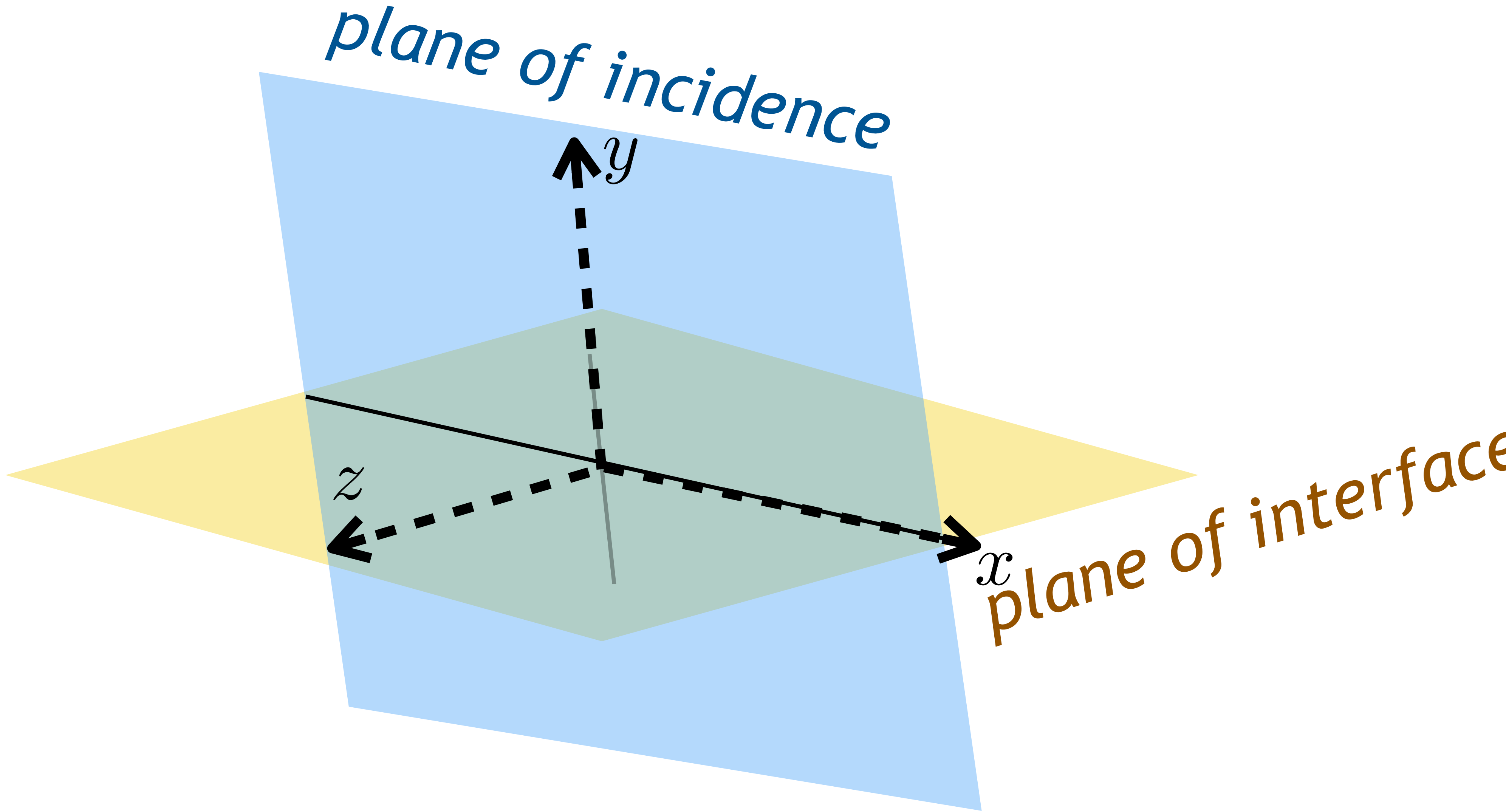
Snell's law of refraction

n is the relative index of refraction.

$$k \sin\theta = k'' \sin\phi$$

$$\frac{\sin\theta}{\sin\phi} = \frac{k''}{k} = \frac{n_2}{n_1} = n$$

Some definitions



- If the electric vector of the incident wave lies in the plane of interface, it is called ***TE* or *transverse electric polarisation***.

- If the magnetic vector of the incident wave lies in the plane of interface, it is called ***TB* or *transverse magnetic polarisation*.**

- Any general case can be treated as a linear superposition of these two cases.

Co-efficients of reflection and transmission

$$r_s = \left[\frac{E'}{E}\right]_{TE} \qquad r_p = \left[\frac{E'}{E}\right]_{TM}$$

$$t_s = \left[\frac{E''}{E}\right]_{TE} \qquad t_p = \left[\frac{E''}{E}\right]_{TM}$$

$$r_s = \frac{\cos\theta - n\cos\phi}{\cos\theta + n\cos\phi}$$

$$r_p = \frac{-n\cos\theta + \cos\phi}{n\cos\theta + \cos\phi}$$

$$n_2/n_1 = n$$

Normal incidence $\theta = \phi = 0$

$$r_s = r_p = \frac{1-n}{1+n}$$

Co-efficients of reflection and transmission

$$r_s = \frac{\cos\theta - n\cos\phi}{\cos\theta + n\cos\phi}$$

$$r_p = \frac{-n\cos\theta + \cos\phi}{n\cos\theta + \cos\phi}$$

$$n_2/n_1 = n$$

Normal incidence $\theta = \phi = 0$

$$r_s = r_p = \frac{1-n}{1+n}$$

$$r_s = \frac{\cos\theta - \sqrt{n^2 - \sin^2\theta}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}}$$

(After using Snell's law)

$$r_p = \frac{-n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}}{n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}}$$

Co-efficients of reflection and transmission

Reflectance is defined as the fraction of the incident light energy that is reflected

$$R_s = |r_s|^2 \qquad r_s = \left[\frac{E'}{E}\right]_{TE}$$

$$R_p = |r_p|^2 \qquad r_p = \left[\frac{E'}{E}\right]_{TM}$$

Total Reflection means

$$R_s = R_p = 1$$

Normal incidence $\theta = \phi = 0$

$$R_s = R_p = \left(\frac{1-n}{1+n}\right)^2$$

For air-glass interface: $n=1.5$

$$R_s = R_p = 0.04$$

(Normal incidence value)

In an optical instrument, with multiple lenses, the loss will become significant, and it can be eliminated by coating anti-reflecting films (MgF_2).

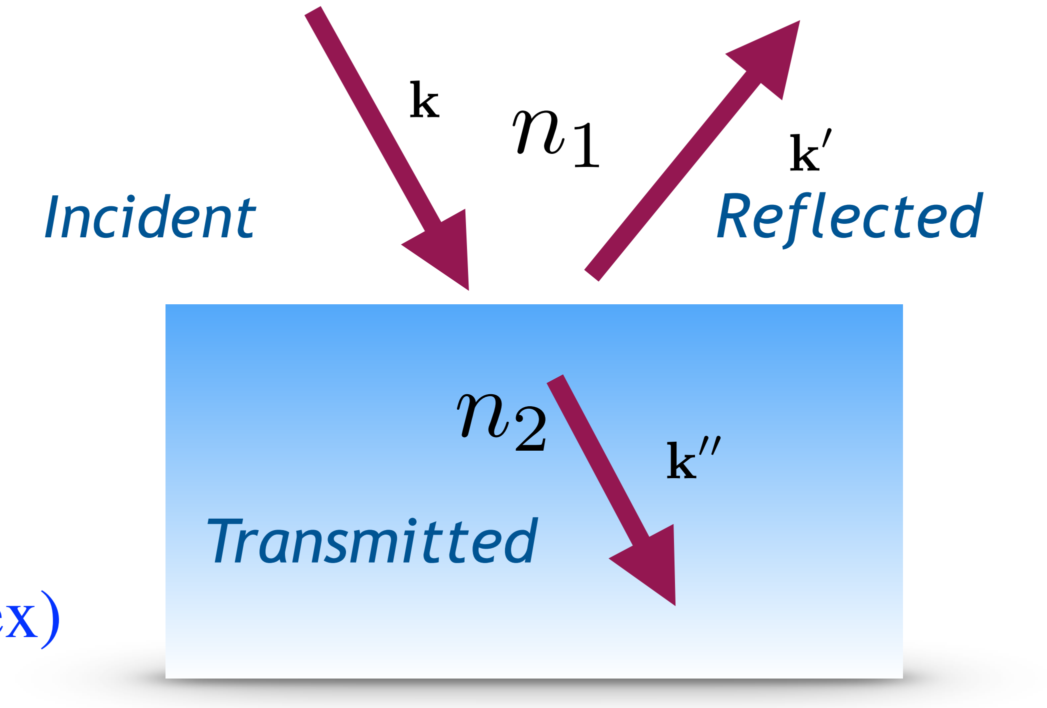
Co-efficients of reflection and transmission

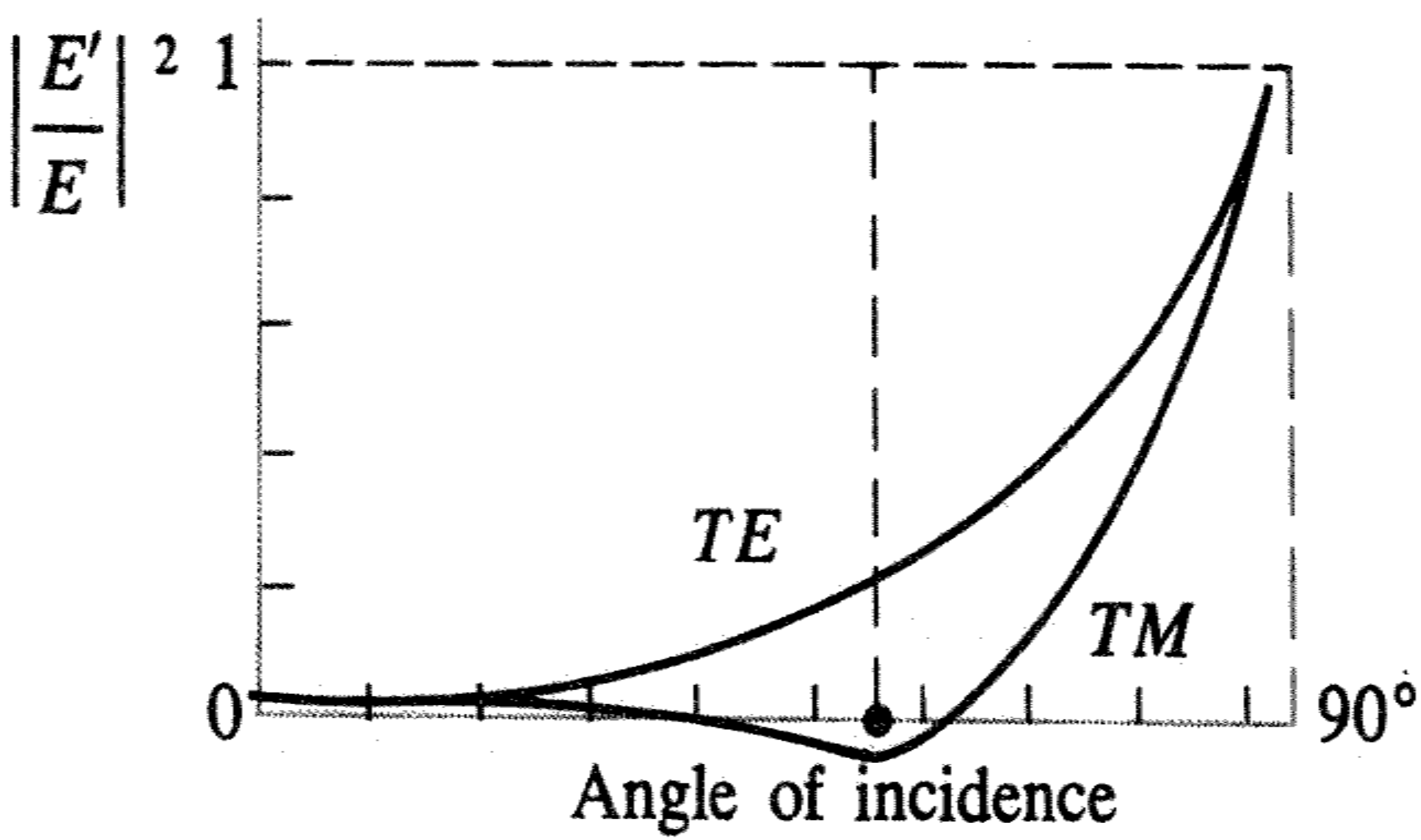
$n_2/n_1 > 1 \implies$ External reflection

(Light comes from medium with less refractive index)

$n_2/n_1 < 1 \implies$ Internal reflection

(Light comes from medium with higher refractive index)





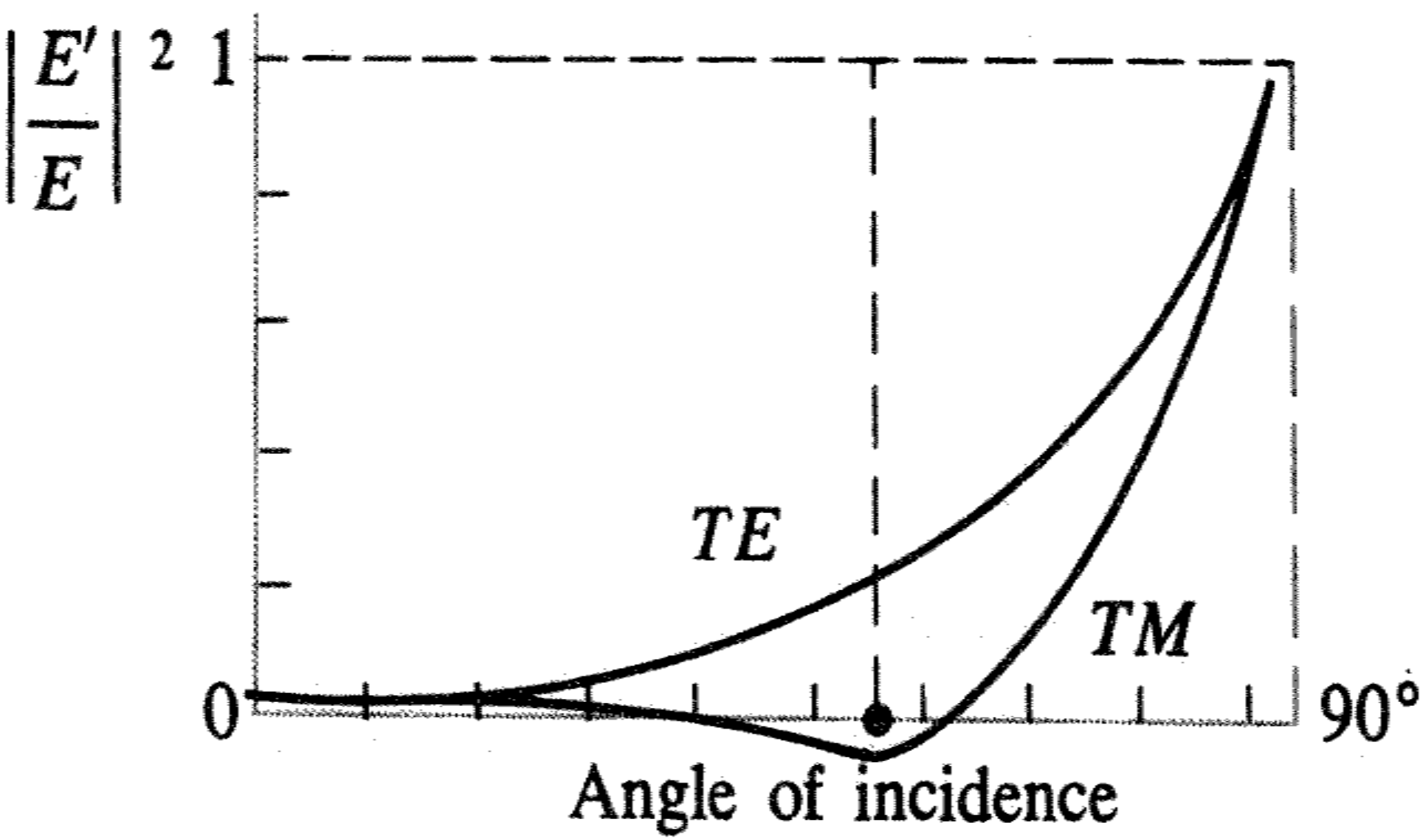
Brewster Angle

$$\tan\theta = n$$

$$r_p = 0$$

Co-efficients of reflection and transmission

By making use of Brewster angle one can generate a linearly polarised (reflected) light from an unpolarised light.



Brewster Angle

$$\tan\theta = n$$

$$r_p = 0$$

Total internal Reflection

$$r_s = \frac{\cos\theta - n\cos\phi}{\cos\theta + n\cos\phi} \Rightarrow \qquad r_s = \frac{\cos\theta - \sqrt{n^2 - \sin^2\theta}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}}$$

(After using Snell's law)

$\sqrt{n^2 - \sin^2\theta} \Rightarrow$ becomes imaginary if :

$$n_2/n_1 < 1 \Rightarrow \text{ Internal reflection}$$

$$\theta > \theta_c = \sin^{-1} n \text{ a critical angle.}$$

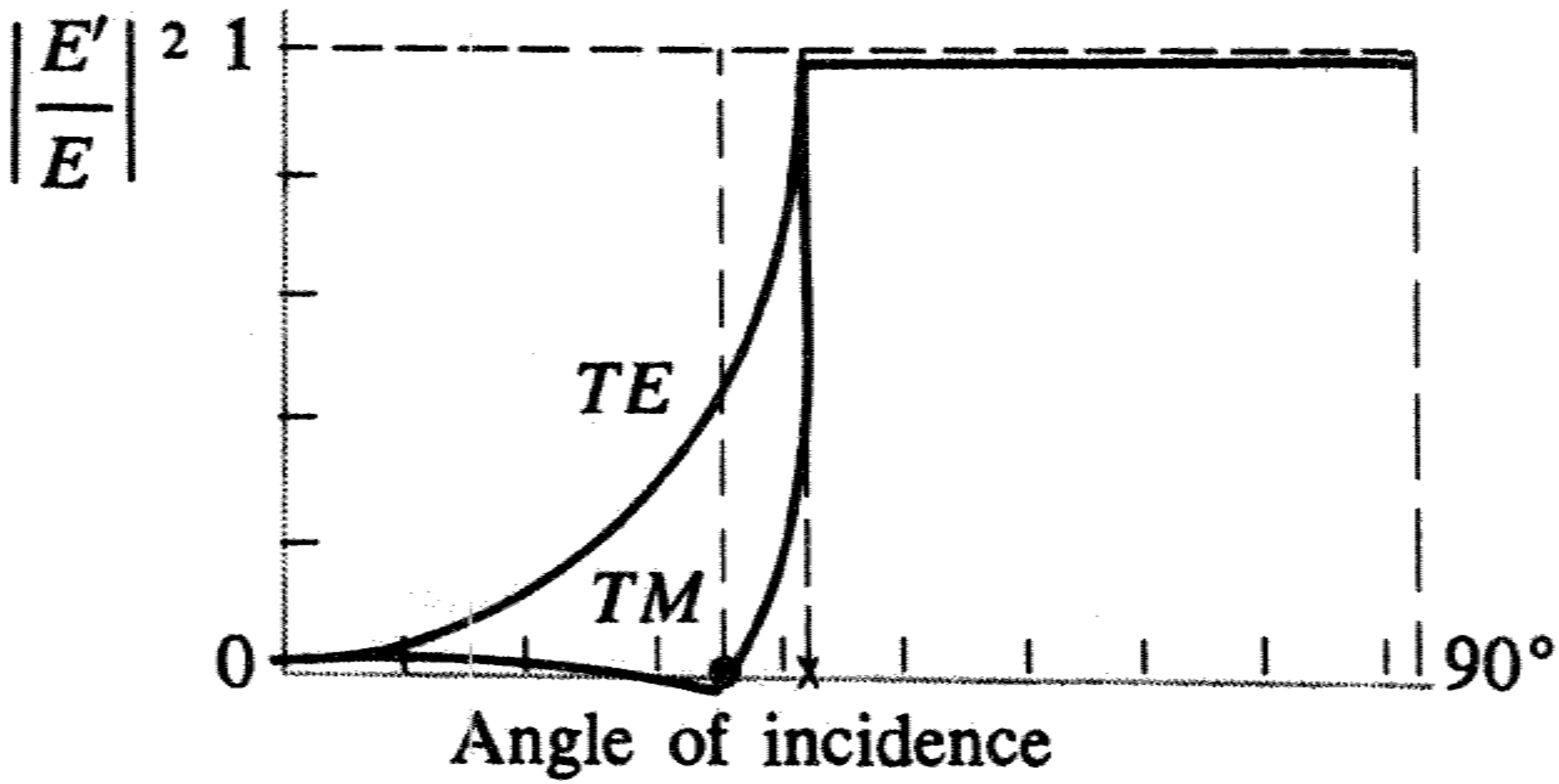
Then it is easy to show that in this case

$$R_s = R_p = 1 \qquad \text{(no transmission)}$$

Total internal Reflection

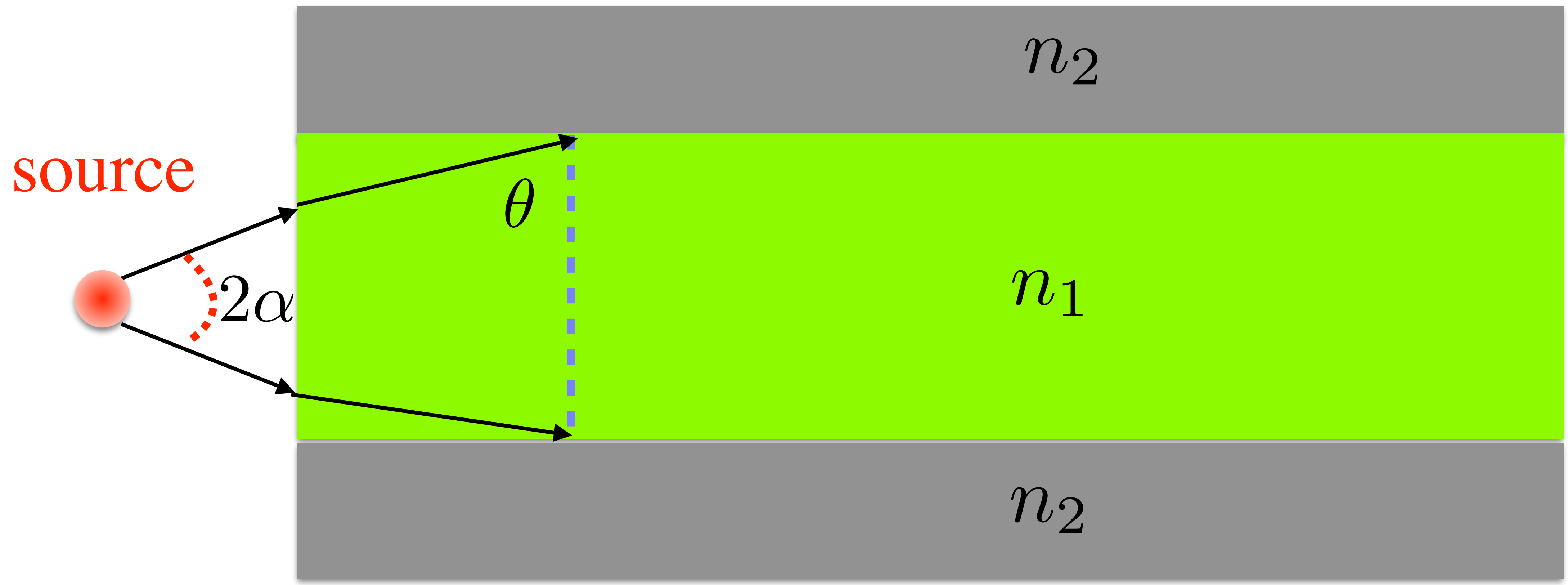
$$r_s = \frac{\cos\theta - n\cos\phi}{\cos\theta + n\cos\phi} \quad \Rightarrow \quad r_s = \frac{\cos\theta - \sqrt{n^2 - \sin^2\theta}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}}$$

(After using Snell's law)



Total internal Reflection (TIR)

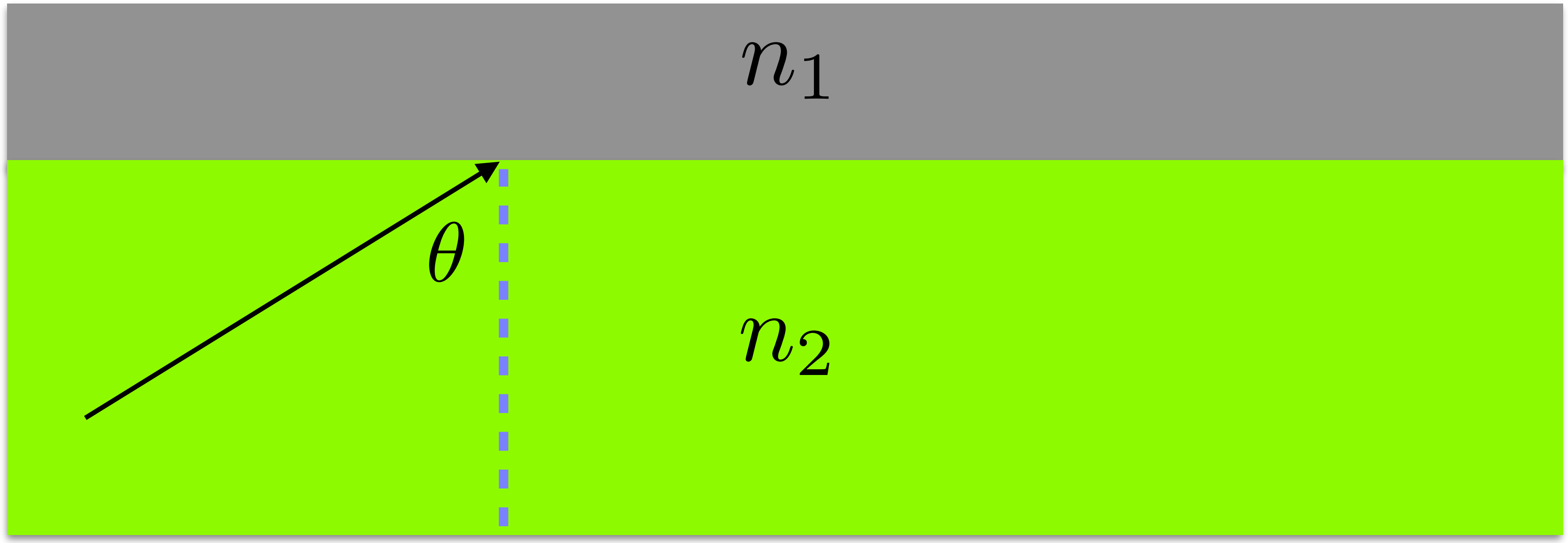
One of the main application of TIR is in the optical wave guides.



$\theta > \theta_c = \sin^{-1} n$ a critical angle.

2α acceptance angle.

Evanescent wave



$$\theta > \theta_c = \sin^{-1} n$$ a critical angle at which TIR happens

Still can I see something on the other side? transmitted????

$$\mathbf{E}_{tr} = \mathbf{E}'' e^{i\left(\mathbf{k}''\cdot\mathbf{r}-\omega t\right)}$$

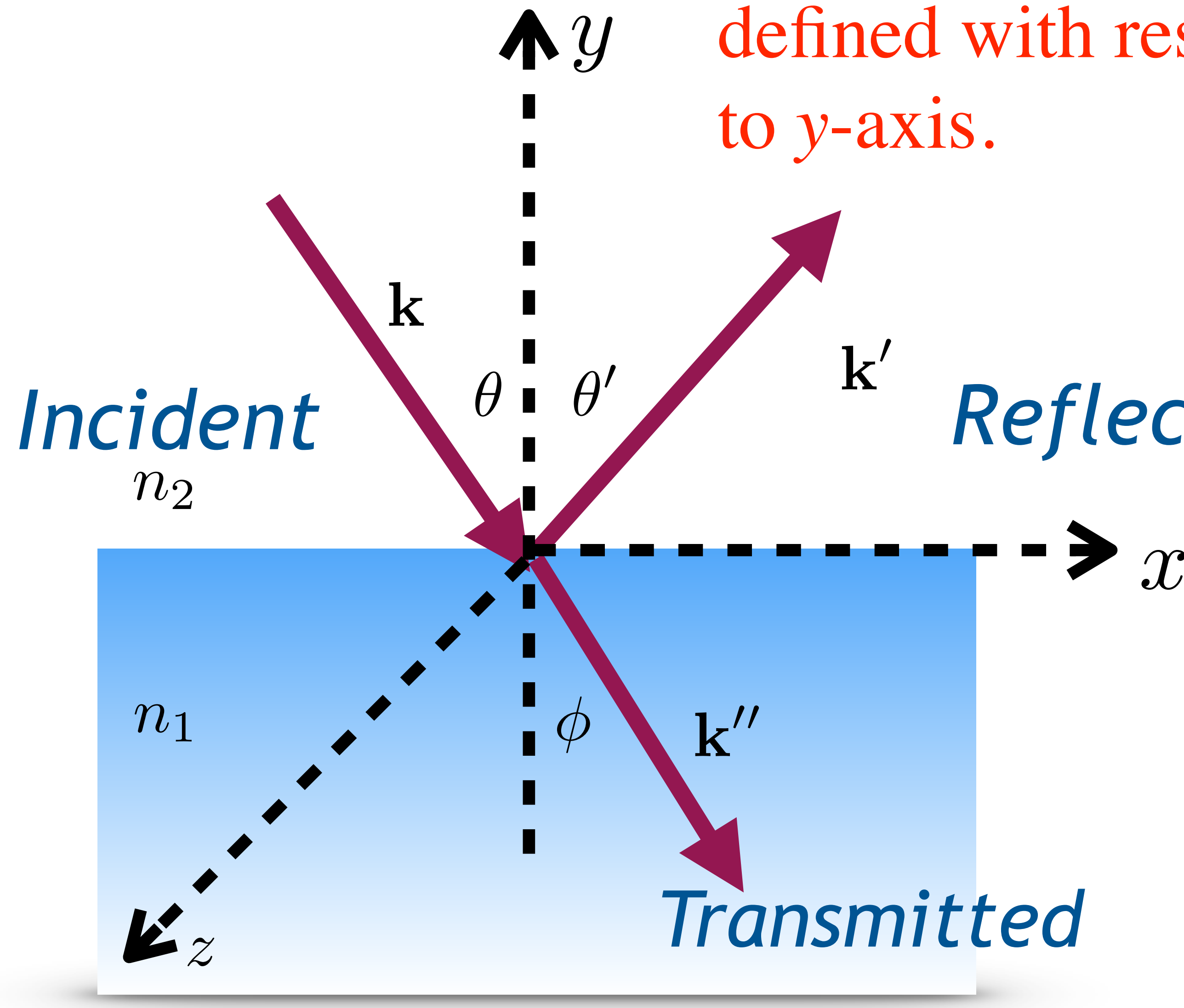
Evanescent wave

$$\mathbf{E}_{tr} = \mathbf{E}'' e^{i\left(\mathbf{k}''\cdot\mathbf{r}-\omega t\right)}$$

The angles are defined with respect to *y*-axis.

Snell's law of refraction

$$\frac{\sin\theta}{\sin\phi} = n$$



$$\mathbf{k}'' \cdot \mathbf{r} = k''x\sin\phi - k''y\cos\phi$$

$$= k''x\sin\phi - ik''y\sqrt{\frac{\sin^2\theta}{n^2} - 1}$$

Evanescent wave

$$\mathbf{k}'' \cdot \mathbf{r} = k'' x \sin\phi - i k'' y \sqrt{\frac{\sin^2\theta}{n^2} - 1}$$

$$\sin\theta > n$$

Total internal reflection

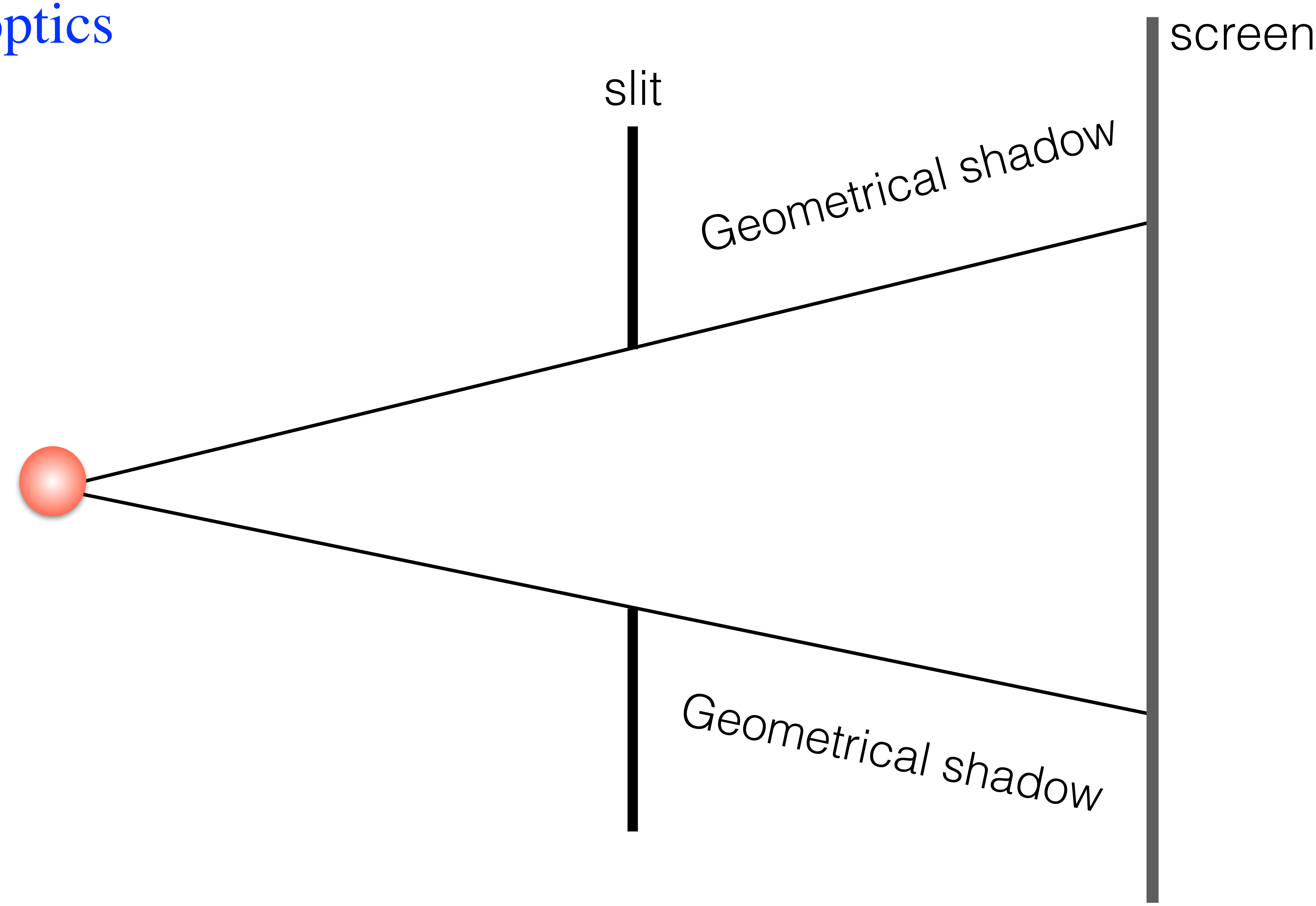
$$\mathbf{E}_{tr} = \mathbf{E}'' e^{-\alpha|y|} e^{i(k_t x - \omega t)}$$

Exponential decay of the transmitted wave inside the medium.

$$\alpha = k'' \sqrt{\frac{\sin^2\theta}{n^2} - 1} \qquad k_t = \frac{k'' \sin\theta}{n}$$

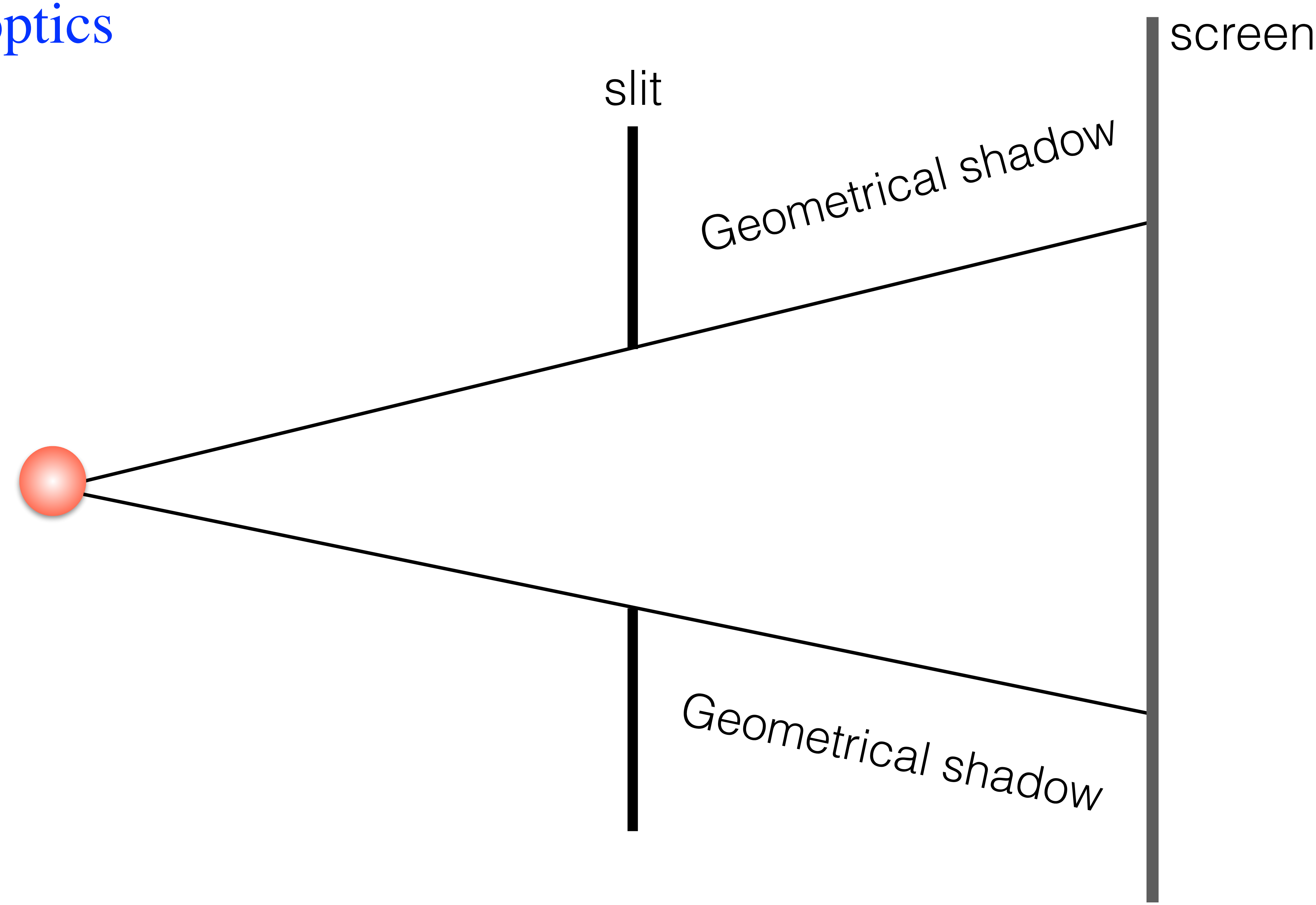
This wave is called Evanescent wave or surface wave.

Ray optics



If the wave length of the light is much small compared to the slit width or the radius of a circular aperture, we have well-defined boundaries.

Ray optics



What happened if I reduce the slit width???

What happened if the wavelength of a light is taken to be zero???

(Geometrical or ray optics)

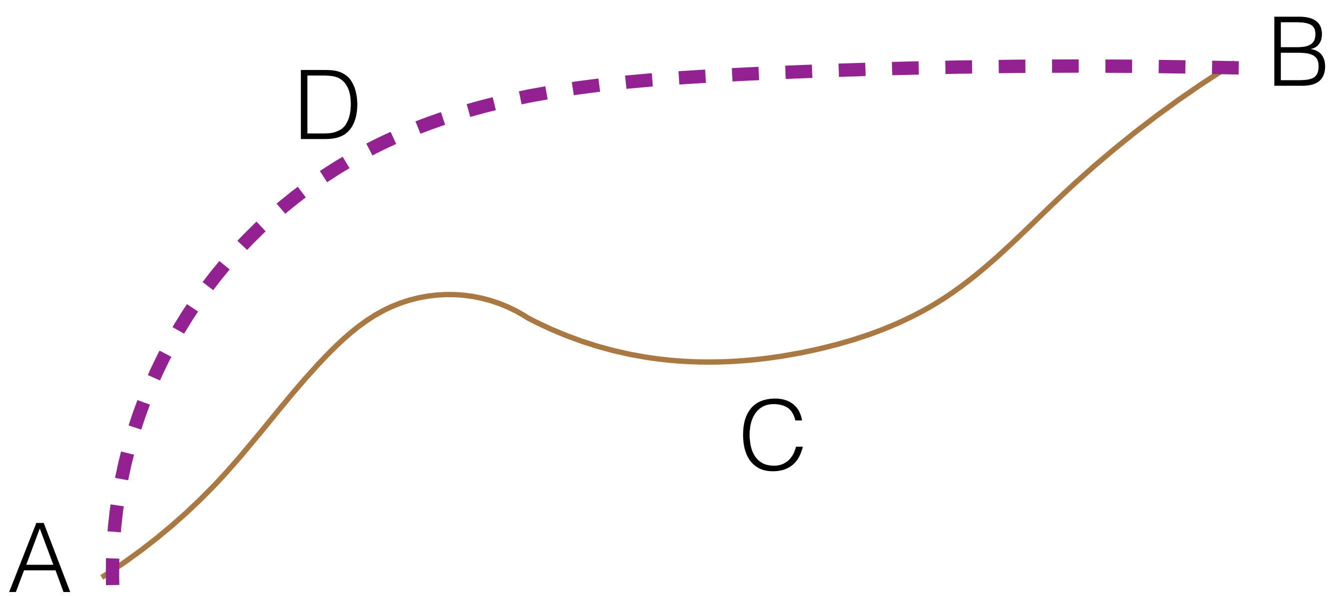
Ray optics

In a homogeneous medium light ray travels in straight line.
(Homogeneous medium means refractive index independent of position)

- In general the path of the ray is determined by the **Fermat's principle.**

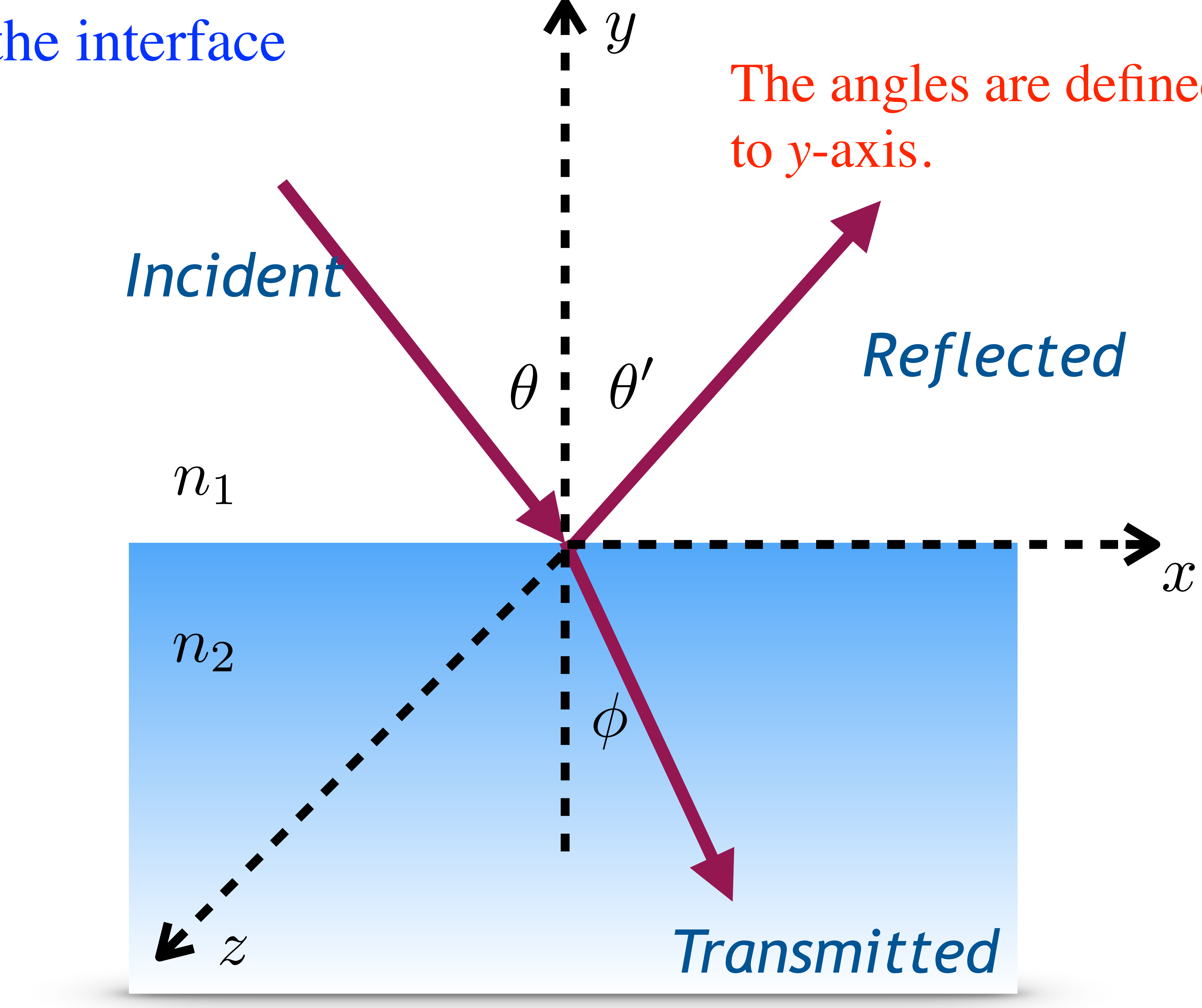
(Similar to principle of least action in Mechanics)

" The actual path between two points taken by a beam of light is the one which takes the least time"



Light at the interface

The angles are defined with respect to y-axis.



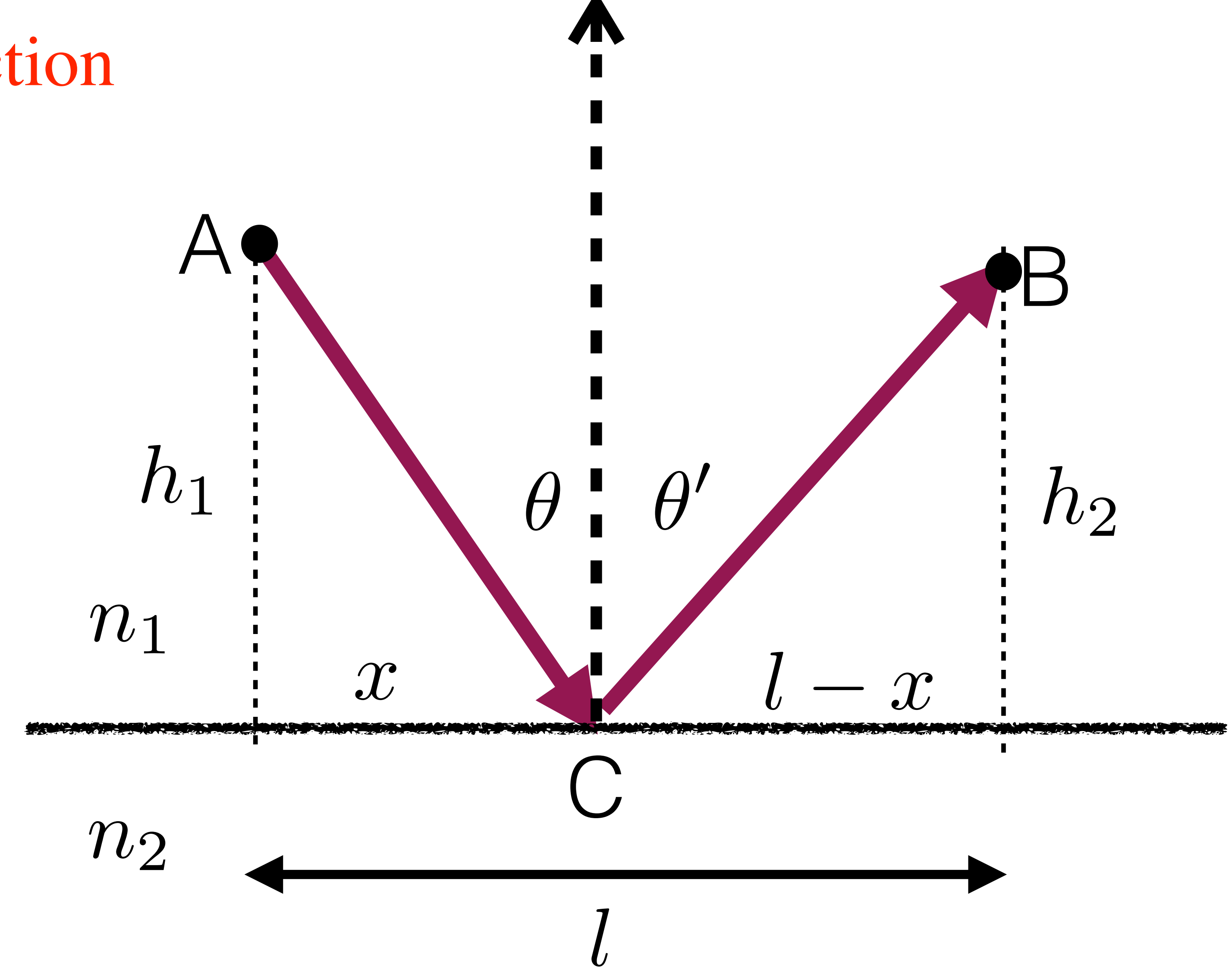
$$\theta = \theta'$$

Law of Reflection

$$\frac{\sin\theta}{\sin\phi} = \frac{n_2}{n_1}$$

Snell's law of refraction

Law of Reflection



Time taken by light to travel from A to B is

$$t = \frac{\sqrt{x^2 + h_1^2}}{c} + \frac{\sqrt{(l-x)^2 + h_2^2}}{c}$$

we can consider different such paths with x being the variable.

Law of Reflection

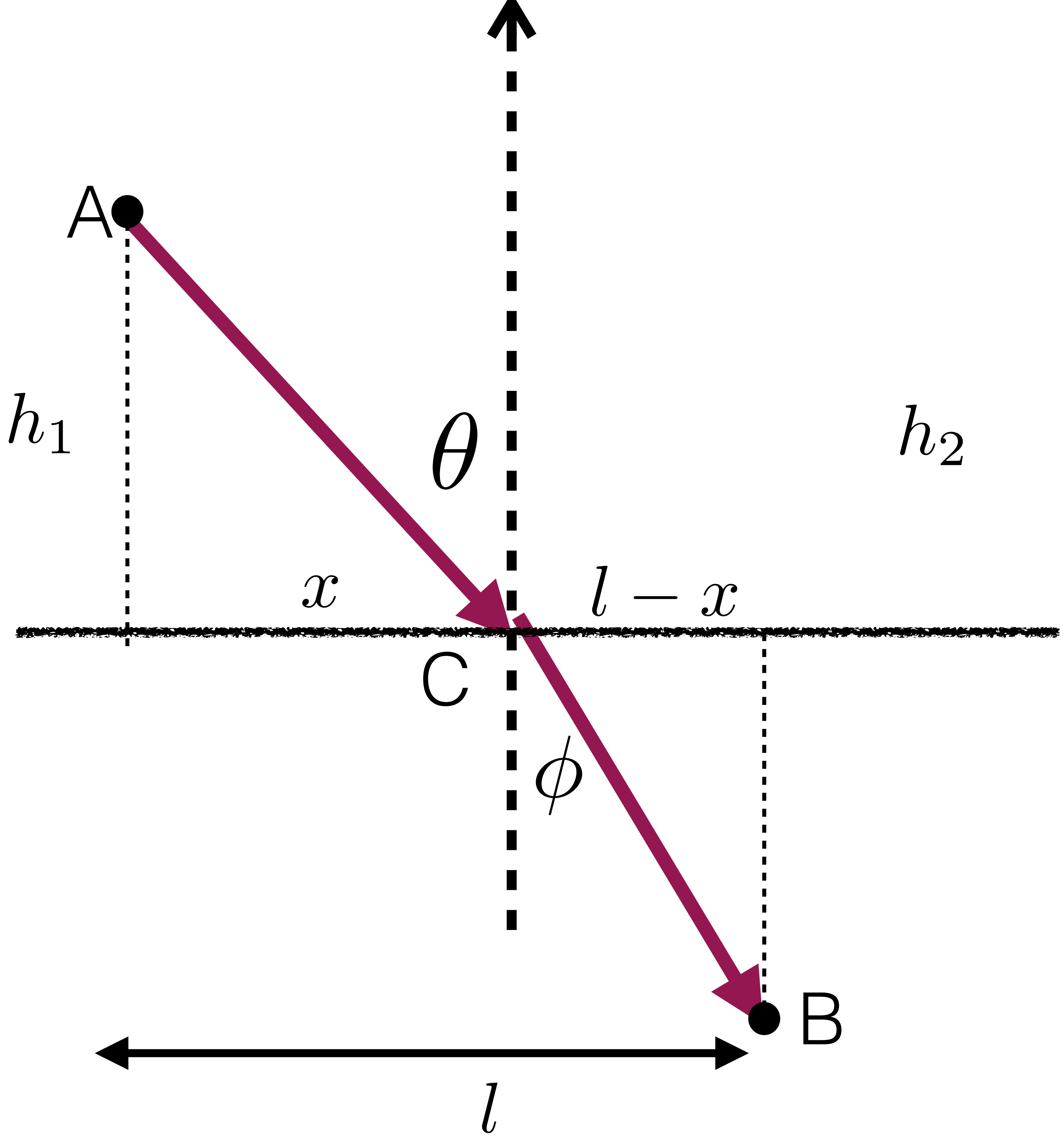
Fermat's principle tells you that for the real path

$$\frac{dt}{dx} = 0 \qquad\qquad t = \frac{\sqrt{x^2 + h_1^2}}{c} + \frac{\sqrt{(l-x)^2 + h_2^2}}{c}$$

$$\frac{x}{c\sqrt{x^2 + h_1^2}} - \frac{l-x}{c\sqrt{(l-x)^2 + h_2^2}} = 0$$

$$\boxed{\sin\theta_1 = \sin\theta_2}$$

Snell's Law



$$t = \frac{\sqrt{x^2 + h_1^2}}{c/n_1} + \frac{\sqrt{(l-x)^2 + h_2^2}}{c/n_2}$$

Snell's Law

Fermat's principle tells you that for the real path

$$\frac{dt}{dx} = 0$$

$$\frac{n_1 x}{\sqrt{x^2 + h_1^2}} - \frac{n_2(l - x)}{\sqrt{(l - x)^2 + h_2^2}} = 0$$

$$\boxed{n_1 \sin\theta = n_2 \sin\phi}$$

If angles are small

$$\frac{\theta}{\phi} = \frac{n_2}{n_1}$$