Runge 2nd Order Method

Major: All Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

Runge-Kutta 2nd Order Method

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Heun's Method

<u>Heun's method</u>

~~Here $a_2 = 1/2$ is chosen~~

$$\cancel{a_1 = \frac{1}{2}}$$

$$\cancel{p_1 = 1}$$

$$\cancel{q_{11} = 1}$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

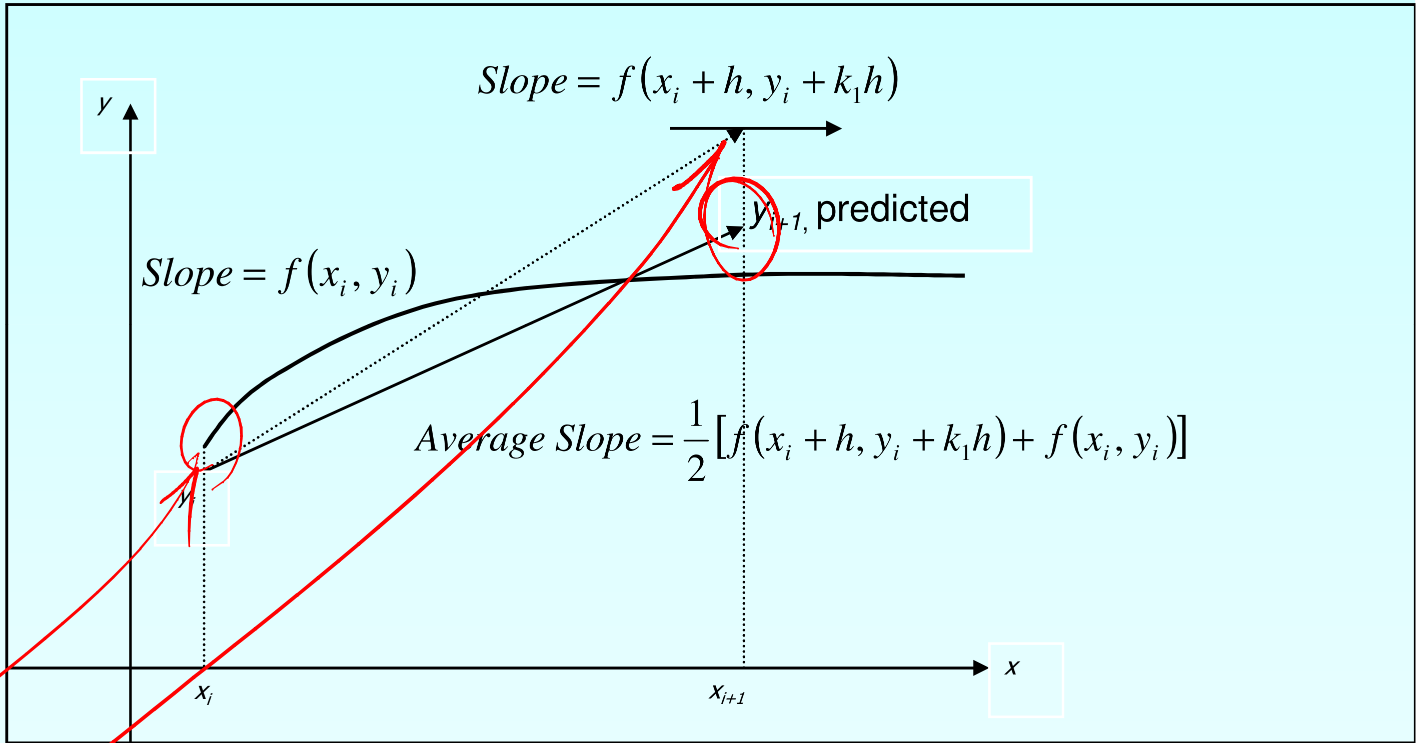


Figure 1 Runge-Kutta 2nd order method (Heun's method)

Runge-Kutta 2nd Order Method

For $$\frac{dy}{dx} = f(x, y),\ y(0) = y_0$$

Runge Kutta 2nd order method is given by

$$y_{i+1} = y_i + \left(\underset{1/2}{a_1} k_1 + \underset{1/2}{a_2} k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \underset{1}{p_1} h, y_i + \underset{1}{q_{11}} k_1 h\right)$$

for Heun's Method

Midpoint Method

Here $a_2 = 1$ **is chosen, giving**

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

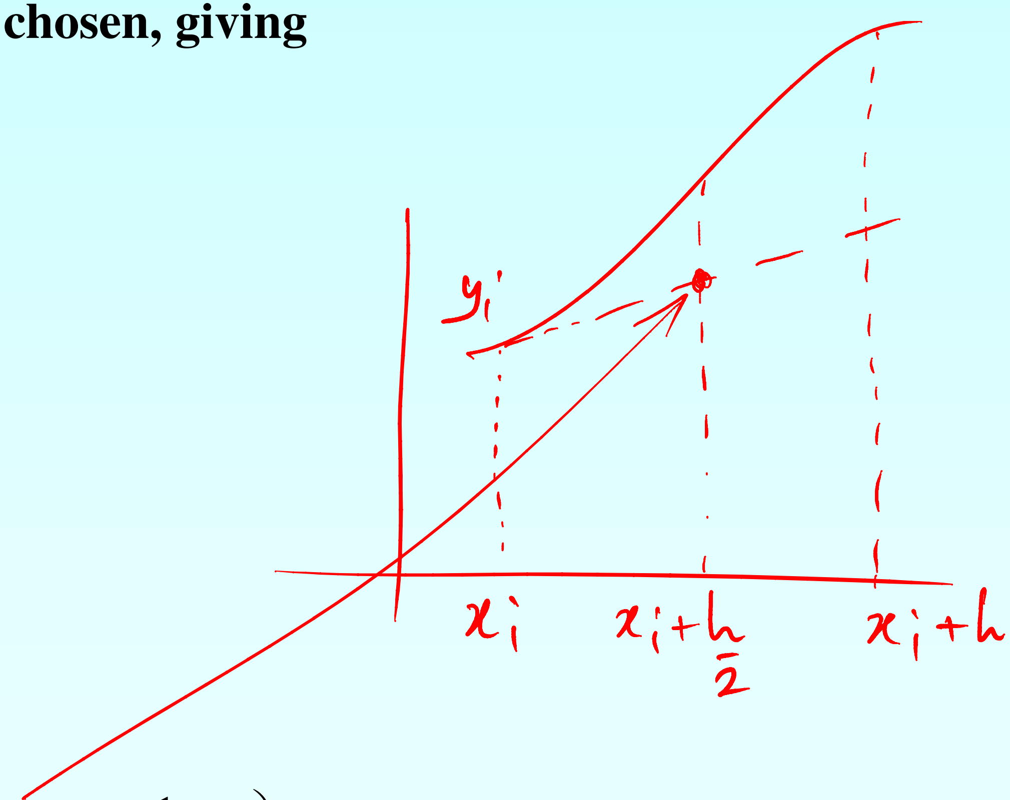
resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1 h\right)$$



Ralston's Method

Here $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

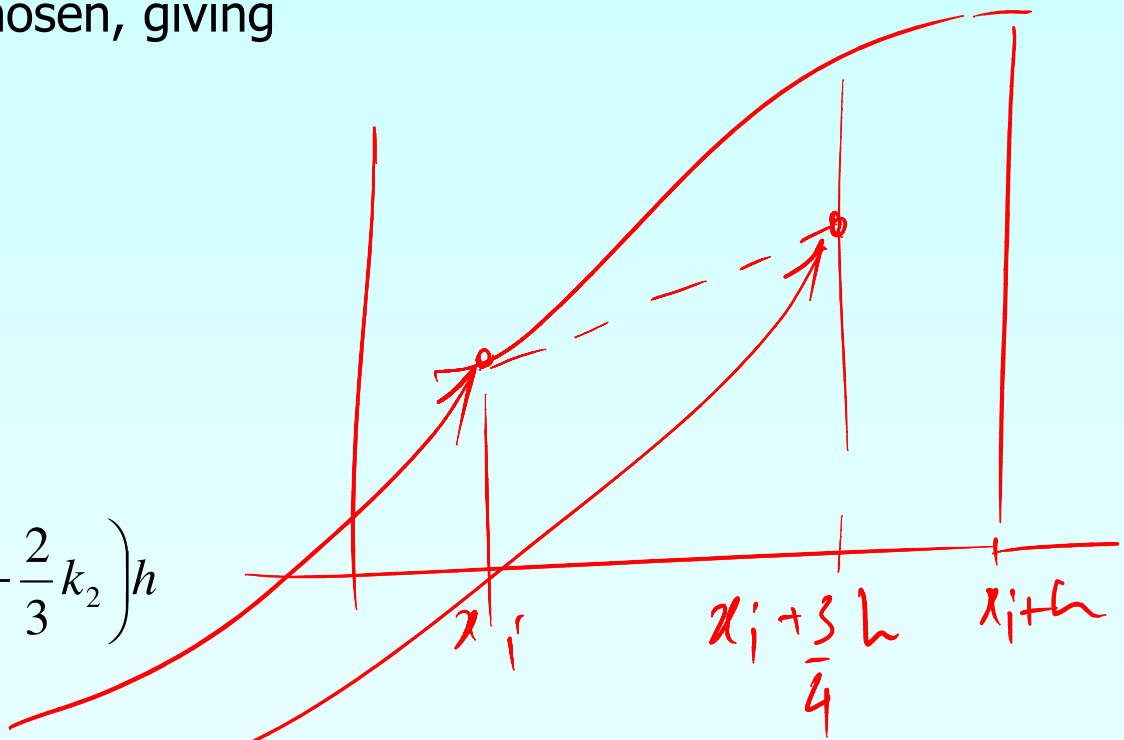
resulting in

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$



How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x},\ y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y,\ y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Heun's method. Assume a step size of $h = 240$ seconds.

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

Solution

Step 1: $i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200K$

$$\begin{aligned}
k_1 &= f(t_0, \theta_o) \\
&= f(0,1200) \\
&= -2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right) \\
&= -4.5579
\end{aligned}$$

$$\begin{aligned}
k_2 &= f(t_0 + h, \theta_0 + k_1 h) \\
&= f(0 + 240, 1200 + (-4.5579)240) \\
&= f(240, 106.09) \\
&= -2.2067\times10^{-12}\left(106.09^4 - 81\times10^8\right) \\
&= 0.017595
\end{aligned}$$

$$\begin{aligned}
\theta_1 &= \theta_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\
&= 1200 + \left(\frac{1}{2}(-4.5579) + \frac{1}{2}(0.017595)\right)240 \\
&= 1200 + (-2.2702)240 \\
&= 655.16K
\end{aligned}$$

Solution Cont

Step 2: $i=1, t_1 = t_0 + h = 0 + 240 = 240, \theta_1 = 655.16K$

$$
\begin{aligned}
k_1 &= f(t_1, \theta_1) \\
&= f(240, 655.16) \\
&= -2.2067\times10^{-12}\left(655.16^4 - 81\times10^8\right) \\
&= -0.38869
\end{aligned}
$$

$$
\begin{aligned}
k_2 &= f(t_1 + h, \theta_1 + k_1 h) \\
&= f(240 + 240, 655.16 + (-0.38869)240) \\
&= f(480, 561.87) \\
&= -2.2067\times10^{-12}\left(561.87^4 - 81\times10^8\right) \\
&= -0.20206
\end{aligned}
$$

$$
\begin{aligned}
\theta_2 &= \theta_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\
&= 655.16 + \left(\frac{1}{2}(-0.38869) + \frac{1}{2}(-0.20206)\right)240 \\
&= 655.16 + (-0.29538)240 \\
&= 584.27K
\end{aligned}
$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta-300}{\theta+300}-1.8519\tan^{-1}\left(0.00333333\theta\right)=-0.22067\times10^{-3}t-2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480)=647.57K$$

Comparison with exact results

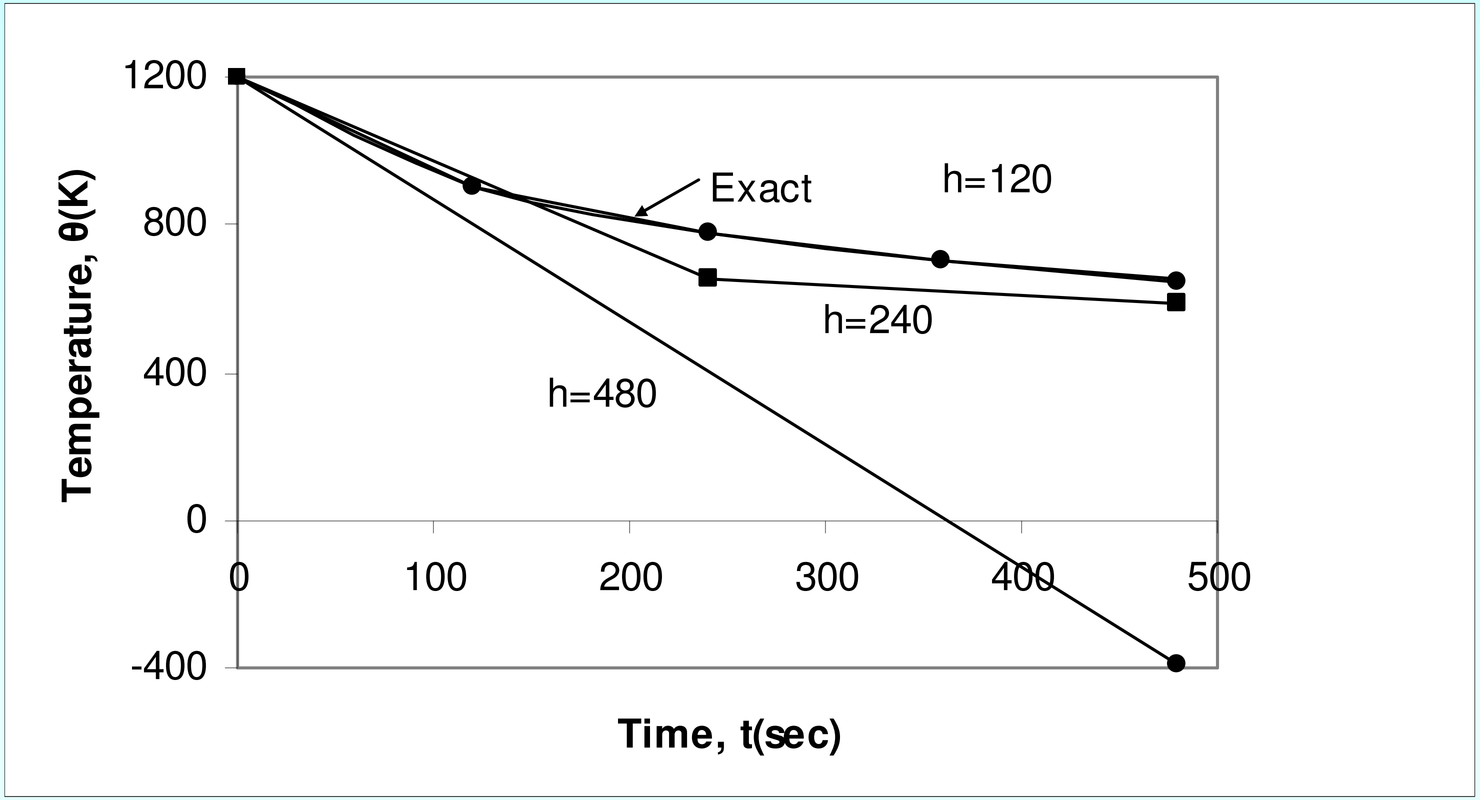


Figure 2. Heun's method results for different step sizes

Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step size, h	$\theta(480)$	E_t	$\|\epsilon_t\|\%$
480	−393.87	1041.4	160.82
240	584.27	63.304	9.7756
120	651.35	−3.7762	0.58313
60	649.91	−2.3406	0.36145
30	648.21	−0.63219	0.097625

$\theta(480) = 647.57\,K$ (exact)

Effects of step size on Heun's Method

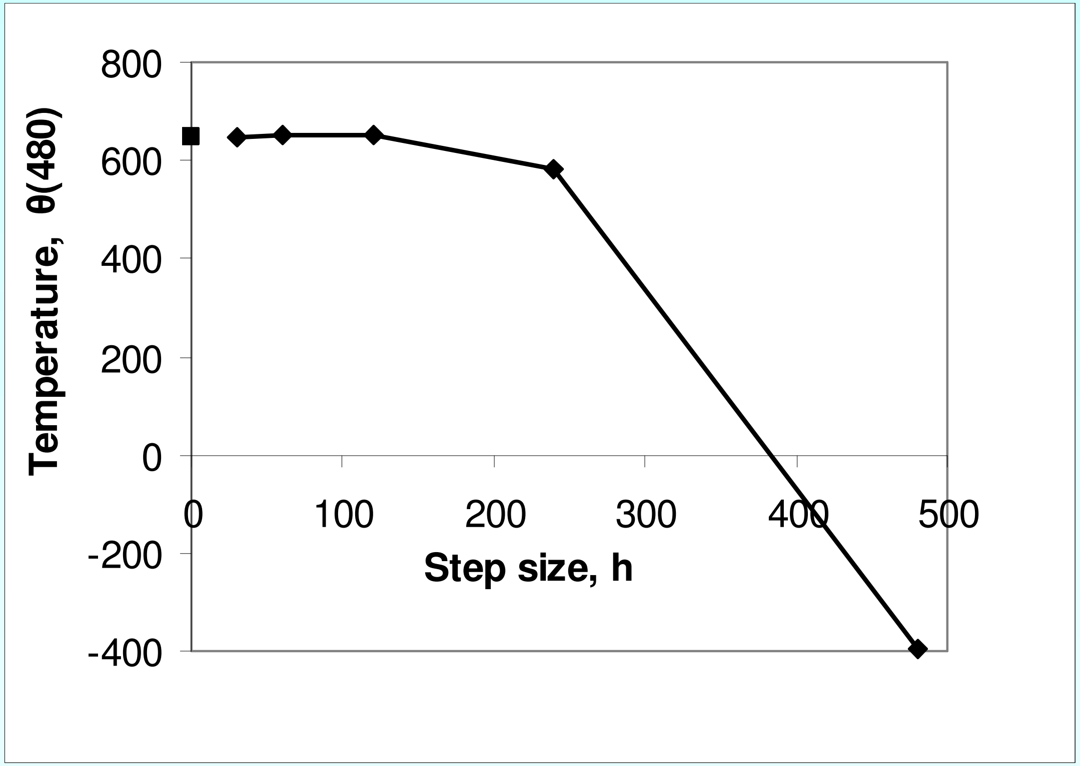


Figure 3. Effect of step size in Heun's method

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\theta(480)$			
	Euler	Heun	Midpoint	Ralston
480	−987.84	−393.87	1208.4	449.78
240	110.32	584.27	976.87	690.01
120	546.77	651.35	690.20	667.71
60	614.97	649.91	654.85	652.25
30	632.77	648.21	649.02	648.61

$$\theta(480) = 647.57K \quad \text{(exact)}$$

Comparison of Euler and Runge-Kutta 2^{nd} Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\lvert\epsilon_t\rvert\%$			
	Euler	Heun	Midpoint	Ralston
480	252.54	160.82	86.612	30.544
240	82.964	9.7756	50.851	6.5537
120	15.566	0.58313	6.5823	3.1092
60	5.0352	0.36145	1.1239	0.72299
30	2.2864	0.097625	0.22353	0.15940

$\theta(480) = 647.57\,K$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

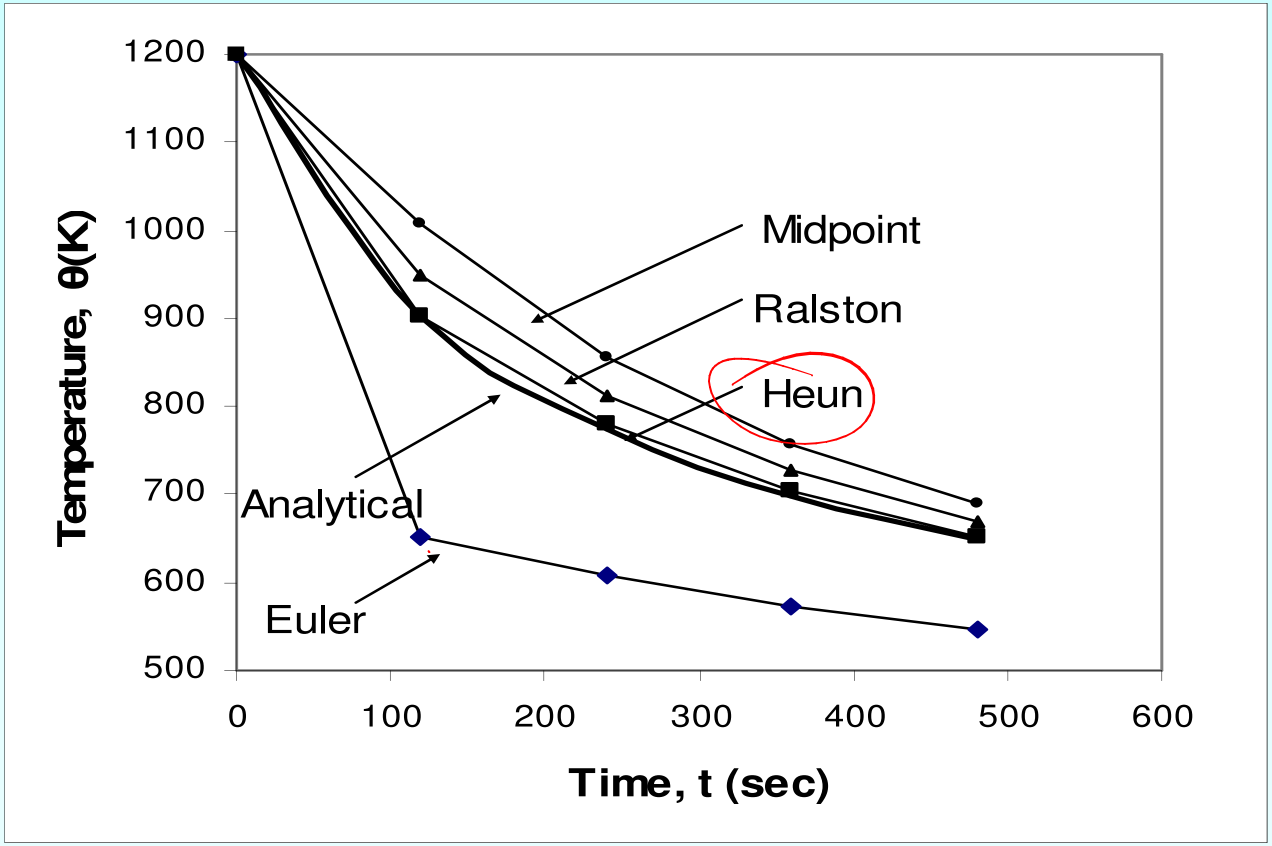


Figure 4. Comparison of Euler and Runge Kutta 2nd order methods with exact results.

Is there a method to this madness?!

First of all note the Taylor expansion in two variables is:

$$\boxed{f(x+h,\ y+k) = f(x,y) + hf_x + kf_y}$$

$$+ \frac{1}{2!}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right)$$

$$+ \frac{1}{3!}\left(h^3 f_{xxx} + 3h^2k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}\right)$$

$$+ \ldots$$

First

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i, y_i} h + \frac{1}{2}\left.\frac{d^2y}{dx^2}\right|_{x_i, y_i} h^2 + O(h^3)$$

Note that $f'(x,y) = f_x + f_y \dfrac{dy}{dx}$ f

thus

$$\boxed{y_{i+1} = y_i + f h + \frac{1}{2} f_x h^2 + \frac{1}{2} f_y f h^2 + O(h^3)} \quad \text{— (1)}$$

Next RK Methods have the form

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) h$$

Notice that $k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$

$$= f + p_1 h f_x + q_{11} k_1 h f_y + O(h^2)$$

(by Taylor expansion)

So

$$y_{i+1} = y_i + \left[a_1 f + a_2 \{f + p_1 h f_x + q_{11} k_1 h f_y + O(h^2)\}\right] h$$

$$= y_i + (a_1 + a_2) h f + a_2 p_1 h^2 f_x + a_2 q_{11} f h^2 f_y + O(h^3) \quad \text{— (2)}$$

Comparing 1 & 2 we have:

$$a_1 + a_2 = 1$$
$$a_2 \, p_1 = 1/2$$
$$a_2 \, q_{11} = 1/2$$

So

	Heun	Midpt	Ralston
a_2	1/2	1	2/3
a_1	1/2	0	1/3
p_1	1	1/2	3/4
q_{11}	1	1/2	3/4

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_2nd_method.html

THE END

http://numericalmethods.eng.usf.edu