ON PARABOLIC BUNDLES ON ALGEBRAIC SURFACES

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Abstract. The aim of this paper is to construct the parabolic version of the Gieseker-Uhlenbeck map from the Gieseker compactification of the moduli space of parabolic stable bundles to the Donaldson-Uhlenbeck compactification of the moduli space of vector bundles with parabolic structures along a divisor with normal crossing singularities on an algebraic surface.

1. **Introduction**

Let X be a smooth projective surface over $\mathbb{C}$. Let D be a divisor with simple normal crossings on X. The study of the moduli space of sheaves with parabolic structures has been done in great generality by Maruyama and Yokogawa in [10]. Their work is a generalisation of the moduli space construction of Mehta and Seshadri [11] for curves.

In [1] we have constructed the Donaldson-Uhlenbeck compactification of the moduli space of parabolic μ-stable(slope stable) bundles.

In this paper our aim is to give an alternate construction of a compactification of the moduli space of parabolic stable sheaves analogous to Gieseker's construction but which proceeds using the orbifold point of view. This was the approach in [1] for the construction of the Donaldson-Uhlenbeck spaces as well.

A few words of justification are needed for the construction of yet another compactification. The principal goal for this approach was an explicit realization the morphism from the Gieseker compactification of the moduli space of parabolic bundles to the Donaldson-Uhlenbeck compactification of moduli of parabolic bundles. This is realized in this approach and opens up possibilities of obtaining topological applications coming from the study of this morphism.

This map can be viewed as a generalisation of the Gieseker-Uhlenbeck map constructed by Jun Li [8] to the case of parabolic bundles. The construction of the map is done in two steps. In the first step we convert the problem of parabolic moduli on X to the Γ–moduli problem on a suitable Kawamata covering Y of X (where Γ is a finite group corresponding to a covering). Secondly, we study the determinant line bundles on the Gieseker compactification of the moduli space of Γ–bundles on Y; we obtain the Gieseker-Uhlenbeck map in the Γ–category or what could be termed the orbifold setting. The invariant direct image construction given by Seshadri-Biswas correspondence 2.14, induces the Gieseker-Uhlenbeck map in the parabolic category on X.

1.1. **Notations.** We fix the following notations. Let $M^{par}(P)$ denote the moduli space of parabolic χ–semistable sheaves on X with parabolic Hilbert polynomial

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P and parabolic datum $\mathbf{s}_*$(see Section 2.3). $M^{par}(\mathbf{c}_*)$ denotes the moduli space of parabolic χ–semistable sheaves of rank r on X of type $\mathbf{c}_* = (\mathcal{P}, k, \mathbf{s}_*)$, where $\mathcal{P}$ the determinant of E_*, k the second Chern class of the underlying sheaf of E_* together with parabolic datum $\mathbf{s}_*$. $M^{par,\mu}(\mathbf{c}_*)$ denotes the moduli space of parabolic μ–semistable sheaves $M^{\boldsymbol{\alpha}}_{k,\mathbf{l},\mathbf{r}}$ of rank r with parabolic structure given by $(\boldsymbol{\alpha}, k, \mathbf{l}, \mathbf{r})$ where

- $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_l)$,
- $\mathbf{l} = (deg(F_1), deg(F_2), ..., deg(F_l))$
- $\mathbf{r} = (rank(F_1/F_2), rank(F_2/F_3), ..., rank(F_l/F_{l+1}))$

In our notation fixing $\mathbf{s}_*$ is equivalent to fixing the tuple $(\boldsymbol{\alpha}, k, \mathbf{l}, \mathbf{r})$.

The main theorem of the paper is:

Theorem 1.1. *Let X be a smooth projective algebraic surface over $\mathbb{C}$. Let Θ_1 be an ample divisor on X. Let D be a normal crossing divisor on X. Let P be a polynomial in $\mathbb{Q}[z]$. We fix $\mathbf{s}_*$ a parabolic datum. Then there is a moduli space $M^{par}(P)$ of parabolic χ-semistable sheaves E_* of rank r on X with parabolic Hilbert polynomial P. Moreover, there is a morphism γ which we term the "Gieseker-to-Uhlenbeck" map for the parabolic bundles:*

$$\gamma : M^{par}(\mathbf{c}_*) \longrightarrow M^{par,\mu}(\mathbf{c}_*)$$

The theorem is an immediate consequence to Theorem 4.14 and Theorem 5.11. Actually we have Theorem 4.14 in a more general setting. Let Y be a smooth projective algebraic variety over $\mathbb{C}$. Let Θ be an ample line bundle on Y. We prove that there is a coarse moduli space M^{Γ} which represents the equivalence classes of pure d-dimensional (Γ, χ)-semistable sheaves on Y.

To construct the moduli space of (Γ, χ)–semi stable sheaves, we work with the Γ-fixed points R^{Γ}(see Section 4) of the Quot scheme $Quot(V \otimes \mathcal{W}, P)$ of semistable coherent sheaves with fixed Hilbert polynomial P(the study of Γ-fixed points is due to Seshadri in the curve case [13], see also [14]). We show that R^{Γ} can be embedded into a suitable Grassmannian variety(see section 4). We use GIT to define semistable points in this Grassmannian variety. Using the embedding we identify the semistable points in R^{Γ}. We prove that the semistable points in the scheme R^{Γ} are in fact the (Γ, χ)–semistable sheaves parametrised by the scheme. Then we use GIT to conclude that a good quotient M^{Γ} of the R^{Γ} exists. This good quotient is indeed the moduli space of (Γ, χ)–semistable sheaves. The moduli space M^{τ} of (Γ, χ)–semistable sheaves of certain local type τ is a Gieseker compactification of the moduli space of parabolic stable bundles on X.

In the end, using the universal property of the categorical quotient, we have the Gieseker-to-Uhlenbeck map γ.

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2. Parabolic bundles

Let X be a connected smooth projective variety of dimension n. Let D be an effective divisor on X. For a coherent sheaf E on X the image of $E \bigotimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$ in E will be denoted by $E(-D)$. The following definition of parabolic sheaf was introduced in [10].

Definition 2.1. [10, Definition 2.3] *Let E be a torsion-free $\mathcal{O}_X$-coherent sheaf on X. A quasi-parabolic structure on E over D is a filtration by $\mathcal{O}_X$-coherent subsheaves*

$$E = F_1(E) \supset F_2(E) \supset \cdots \supset F_l(E) \supset F_{l+1}(E) = E(-D)$$

The integer l is called the length of the filtration. A parabolic structure is a quasi-parabolic structure, as above, together with a system of weights $\{\alpha_1, \cdots, \alpha_l\}$ such that

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{l-1} < \alpha_l < 1$$

where the weight α_i corresponds to the subsheaf $F_i(E)$.

We shall denote the parabolic sheaf defined above by (E, F_*, α_*). When there is no risk of confusion it will be denoted by E_*.

Remark 2.2. In general one associates a tuple $\{\alpha_{1\lambda}, \cdots, \alpha_{l\lambda}\}$ to each component D_λ of the divisor D. We assume that for all components D_λ the same tuple $\{\alpha_1, \cdots, \alpha_l\}$ is associated.

For a parabolic sheaf (E, F_*, α_*), define the following filtration $\{E_t\}_{t\in\mathbb{R}}$ of coherent sheaves on X parametrized by $\mathbb{R}$:

$$E_t := F_i(E)(-[t]D) \tag{2.1}$$

where $[t]$ is the integral part of t and $\alpha_{i-1} < t - [t] \leq \alpha_i$, with the convention that $\alpha_0 = \alpha_l - 1$ and $\alpha_{l+1} = 1$.

Remark 2.3. Let F be a proper subsheaf of E such that the quotient is torsion-free, then there is a canonical filtration $\{F_t\}_{t\in\mathbb{R}}$ of coherent sheaves on X where $F_t = E_t \cap F$. The parabolic structure induced by this filtration to the sheaf F is called the induced parabolic structure on F.

A *homomorphism* from the parabolic sheaf (E, F_*, α_*) to another parabolic sheaf (E', F'_*, α'_*) is a homomorphism from E to E' which sends any subsheaf E_t into E'_t, where $t \in [0, 1]$ and the filtrations are as above.

If the underlying sheaf E is locally free, then E_* will be called a parabolic vector bundle.

We recall the "covering lemma" of Kawamata: Let X be a smooth projective variety over $\mathbb{C}$. Let $D = \sum_{i=1}^{d} D_i$ be the decomposition of the normal crossing divisor D into irreducible components. Let N be an integer.

'The covering lemma' of Kawamata ([7, Theorem 1.1.1] [6, Theorem 17]) says that there is a smooth projective variety Y over $\mathbb{C}$ and a Galois covering morphism $p : Y \longrightarrow X$ such that the reduced divisor $\tilde{D} := (p^*D)_{red}$ is a normal crossing divisor on Y and furthermore, $p^*D_i = k_iN \cdot (p^*D_i)_{red}$, where $k_i, 1 \leq i \leq d$ are positive integers. Define $\tilde{D}_i := (p^*D_i)_{red}$; so, $p^*(D_i) = k_iN\tilde{D}_i$. Let Γ denote the Galois group for the covering map p(See [9, Proposition 4.1.12]).

2.1. **On Γ bundles.** Let Y be a smooth projective variety over $\mathbb{C}$. Let Γ be a finite subgroup of $Aut(Y)$ the group of automorphisms of Y. We assume that the group Γ acts on Y such that the projection $p : Y \longrightarrow Y/\Gamma$ is a ramified covering morphism with $X := Y/\Gamma$ a smooth projective variety. Note that this is a finite morphism. Let $y \in Y$, then we denote the isotropy group of Γ for y by Γ_y. From now onwards we fix this action of Γ on Y and hence call Y a Γ–variety.

Then it is clear that the structure sheaf $\mathcal{O}_Y$, the sheaf of regular functions on Y gets a Γ–action coming from the action on Y.

Definition 2.4. *(see [2]) Let $\mathcal{E}$ be a coherent sheaf of $\mathcal{O}_Y$-modules on Y such that the action of the group Γ on Y lifts to an action of Γ on $\mathcal{E}$, then it is called a Γ–sheaf. This means that Γ acts on the total space of stalks of $\mathcal{E}$, and the automorphism of the space of stalks for the action of any $\gamma \in \Gamma$ is a coherent sheaf isomorphism between $\mathcal{E}$ and $\gamma^*(\mathcal{E})$, where $\gamma : Y \longrightarrow Y$ denotes the corresponding morphism of Y induced by the action of Γ on Y for each $\gamma \in \Gamma$.*

For reference we recall the following theorem:

Theorem 2.5. [1, Theorem 7.1] *Let X be a smooth projective surface and let $p : Y \longrightarrow X$ be a Kawamata covering of X. Let Θ be a pull-back of a very ample divisor Θ_1 on X.*

Let $\phi' : Y \longrightarrow \mathbb{P}^n_{\mathbb{C}}$ be the closed embedding induced by Θ. Then there exists a Γ–hyperplane $Z \subset \mathbb{P}^n_{\mathbb{C}}$, not containing Y, such that the scheme $Z \cap Y$ is regular at every point. Furthermore, the set of hyperplanes with this property forms an open dense subset of $\mid \Theta \mid^{\Gamma}$.

From now on we fix the line bundle Θ as a polarisation on Y. We define the Gieseker type semistability of a Γ–bundle with respect to Θ on Y. For Γ trivial the Gieseker semistability notion is defined in [3].

We denote $\mathcal{O}(n\Theta)$ by $\mathcal{O}(n)$. Let $\mathcal{E}$ be a coherent sheaf on X. The Hilbert polynomial for $\mathcal{E}$ is defined by the condition that $P(\mathcal{E}, m) = dim(H^0(Y, \mathcal{E}(m)))$ for $n \gg 0$. Let d denote the dimension of the support of $\mathcal{E}$. Then d is the degree of the polynomial $P(\mathcal{E}, m)$. Write

$$P(\mathcal{E}, m) = \frac{rm^d}{d!} + \frac{am^{d-1}}{(d-1)!} + \cdots .$$

Then r is the rank of the sheaf $\mathcal{E}$ and $\mu(\mathcal{E}) = \frac{a}{r}$ is the slope of $\mathcal{E}$. The quotient $p(\mathcal{E}, m) := \frac{P(\mathcal{E},m)}{r}$ is called the normalised Hilbert polynomial of $\mathcal{E}$.

Remark 2.6. We recall that a coherent sheaf $\mathcal{E}$ on Y is pure sheaf of dimension d if the dimension of the support of $\mathcal{E}$ is d and if for every subsheaf $\mathcal{F} \neq 0$ of $\mathcal{E}$, the dimension of the support of $\mathcal{F}$ is d. Torsion-free sheaves are supported on Y and they are pure sheaves of dimension $dim(Y)$.

Definition 2.7. *A Γ–torsion free sheaf E on Y is said to be (Γ, χ)-semistable or Gieseker-Γ–semistable if for every Γ–subsheaf F of E the following inequality holds:*

$$p_F(n) \leq p_E(n)$$

for sufficiently large n. We say E is (Γ, χ) stable if the above inequality is strict.

Remark 2.8. When Γ is trivial, the sheaf $\mathcal{E}$ is χ-semistable(stable) if the above inequality is satisfied.

Remark 2.9. Let $\mathcal{E}$ be a coherent Γ–sheaf. Then $\mathcal{E}$ is (Γ, χ)-semistable if it is pure and if for every Γ–subsheaf $\mathcal{F}$ of $\mathcal{E}$ we have

$$r(\mathcal{E})P(\mathcal{F}, m) \leq r(\mathcal{F})P(\mathcal{E}, m)$$

for sufficiently large m. For stable bundles the inequality is strict.

Remark 2.10. Let V be a pure d-dimensional Γ–sheaf of rank r on Y. The Harder-Narasimhan filtration for V is the unique filtration

$$0 \subset V_0 \subset V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_l = V$$

of Γ–subsheaves V_i of V such that V_i/V_{i-1} for $i \in [1, l]$ are (Γ, χ)-semistable sheaves with normalised Hilbert polynomials $p(V_i/V_{i-1})(n)$ strictly decreasing as i increases for large n(cf. [15, Lemma 3.1]).

Definition 2.11. *Let V be a (Γ, χ)-semistable pure d-dimensional sheaf of rank r on Y. A Jordan-Hölder filtration for V is an increasing sequence*

$$V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_n = V$$

of pure d-dimensional Γ–subsheaves V_i of V such that $V_1, V_2/V_1, \cdots V_n/V_{n-1}$ are (Γ, χ)-stable sheaves with $p(V_i/V_{i-1})(n) = p(V)(n)$. The sheaf

$$V_1 \oplus V_2/V_1 \oplus \cdots V_n/V_{n-1}$$

is denoted by $gr^{\Gamma}_{\chi}(V)$ and it is called the associated graded object of V associated to a Jordan-Hölder filtration. The sheaf $gr^{\Gamma}_{\chi}(V)$ is unique.

Remark 2.12. The existence of a Jordan-Hölder filtration can be proved by induction on the rank of V. It is easy to see that $gr^{\Gamma}_{\chi}(V)$ is independent of the choice of a Jordan-Hölder filtration. For details in the case when Γ is *trivial* see [15, Page 90]. It is easy to see that these results generalise for the Γ–case as well.

Lemma 2.13. *A pure d-dimensional Γ–sheaf $\mathcal{E}$ on Y is (Γ, χ)-semistable if and only if it is χ-semistable on Y.*

Proof: If $\mathcal{E}$ is χ-semistable then it is clear that $\mathcal{E}$ is (Γ, χ)-semistable. Conversely, suppose that $\mathcal{E}$ is (Γ, χ)-semistable but not χ-semistable. Then there is the Harder-Narasimhan filtration $0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$ for $\mathcal{E}$. Since the filtration is canonical it is invariant under the automorphisms of $\mathcal{E}$. Hence the elements $\mathcal{E}_i$ of the filtration are Γ–subsheaves of $\mathcal{E}$. Hence each $\mathcal{E}_i/\mathcal{E}_{i-1}$ is (Γ, χ)-semistable. Therefore we have a non-trivial Harder-Narasimhan filtration of the Γ–sheaf $\mathcal{E}$ which contradicts the (Γ, χ)-semistability of $\mathcal{E}$.

q.e.d

2.2. **Some remarks on the category of parabolic sheaves and of Γ–sheaves.** Let X be a smooth projective variety and D be a normal crossing divisor in X and let Θ_1 be an ample divisor in X. Let $\mathcal{P}_{X/D}$ denote the category of parabolic sheaves and their parabolic homomorphisms with parabolic structure over the divisor D. For any coherent sheaf E we can associate a special parabolic structure as

$$E_\alpha = E \otimes \mathcal{O}_X(-mD)$$

for $\alpha \in (m-1, m]$. Then the category of $\mathcal{O}_X$–modules denoted by the $\mathcal{M}_X$ is a full subcategory of the category $\mathcal{P}_{X/D}$. The category $\mathcal{P}_{X/D}$ is an abelian category with enough injectives [17, Proposition 1.1]. For our purpose we assume that the weights are rational with common multiple of $1/N$ where N is an integer. For example $\alpha_i = k_i/N$ where k_i are all integers. The category of parabolic coherent sheaves $P_{X/D}$, is an abelian subcategory of $\mathcal{P}_{X/D}$. The category of parabolic torsion free sheaves is a full subcategory of $\mathcal{P}_{X/D}$ but it is not an abelian category. In the exact sequence of parabolic sheaves

$$0 \longrightarrow E_* \longrightarrow F_* \longrightarrow G_* \longrightarrow 0$$

if any two of them are coherent then the other is a coherent sheaf. In particular F_* is torsion free if both E_* and G_* are torsion free sheaves.

For an integer $N \geq 2$, let $\mathcal{P}(X, D, N) \subseteq \mathcal{P}_{X/D}$ denote the subcategory consisting of all parabolic torsion free sheaves all of whose parabolic weights are multiples of $1/N$.

Let $\mathcal{V}_\Gamma(Y)$ denote the category of Γ–torsion free sheaves on Y. Let $\mathcal{V}_\Gamma^D(Y, N)$ denote the subcategory of $\mathcal{V}_\Gamma(Y)$ consisting of all Γ–torsion free sheaves W over Y satisfying some topological conditions which are induced by the conditions on the parabolic category. For details see [1, Section 2.4.1].

Let W be a Γ-sheaf on Y. Then the invariant direct image sheaf $p_*^\Gamma(W)$ is a sheaf on X which comes with a parabolic structure. In fact the parabolic structure can be written as follows: Let W be a Γ-torsion free sheaf on Y. Let $\tilde{D} \subset \hat{D}$ be the reduced divisor given by the union of all those components of $\hat{D}$ for which the action of the isotropy group of every point of the component is nontrivial on the fibre of W. Note that for a component of $\hat{D}$, it may happen that, at special points of the component, the action of the isotropy subgroup is nontrivial, but at a general point, the action is trivial. Such a component is not included in $\tilde{D}$. Let $D := p(\tilde{D})$ and $D = \sum_{\lambda=1}^h D_\lambda$ as a sum of irreducible components of D. We define $\tilde{D}_\lambda := p^*(D_\lambda)_{red}$. Let $n_\lambda \in \mathbb{N}$ such that $p^* D_\lambda = n_\lambda \cdot \tilde{D}_\lambda$ so that $\tilde{D} = \sum_{\lambda=1}^h \tilde{D}_\lambda$.

Since $\tilde{D}_\lambda$ as a subset of Y is invariant under the action of Γ on Y, for any $k \in \mathbb{Z}$, there is a natural Γ-structure on the line bundle $\mathcal{O}_Y(k\tilde{D}_\lambda)$. Then the filtration $\{E_t\}_{t\in\mathbb{R}}$ defined by

$$E_t := (p_*(W \otimes \mathcal{O}_Y(\sum_{\lambda=1}^h [-t.n_\lambda]\tilde{D}_\lambda)))^\Gamma$$

is a parabolic torsion free sheaf on X with parabolic structure defined on a divisor $D := p(\tilde{D})$. This follows from the work of Biswas [2]. This correspondence is one-one and it is true for any smooth projective variety. Again by using Biswas ideas, we obtain the Γ–sheaf W on Y associated to any parabolic torsion free sheaf E_* on X. We see that the sheaves obtained by this way satisfy the condition that the they are of local type τ since the local type is induced by the parabolic structure.

Remark 2.14. The Γ–invariant direct image functor p_*^Γ is an equivalence of categories called Seshadri-Biswas correspondence

$$p_*^\Gamma : \mathcal{V}_\Gamma^D(Y, N) \longrightarrow \mathcal{P}(X, D, N)$$

Definition 2.15. *Following Seshadri* [13, page 161] *we call the* Γ*–torsion free sheaves* E *in* $\mathcal{V}_\Gamma^D(Y, N)$ torsion free sheaves of fixed local Γ–bundle type τ.

Let L be a parabolic line bundle on X with a parabolic structure on D with a parabolic weight $\alpha = m/N$ a rational number. Then by Seshadri-Biswas correspondence there is a Γ–line bundle $\tilde{L}$ on Y such that $p_*^\Gamma(\tilde{L}) = L$ whose local type τ is given by the weight α. Indeed,

$$\tilde{L} = p^*(L) \otimes \mathcal{O}_Y(\sum_{i=1}^{d} k_i m \tilde{D}_i). \tag{2.2}$$

where $\sum_{i=1}^{d} k_i N \tilde{D}_i$ is the pullback of D on Y.

From this we get $c_1(\tilde{L}) = c_1(p^*(L)) + \alpha c_1(p^*(D))$ which is equivalent to $c_1(\tilde{L}) = p^*(c_1(L) + \alpha c_1(D))$.

Lemma 2.16. *(Splitting principle for Γ–vector bundles on Y) Let E be a Γ-vector bundle of rank r on Y. Let $A(Y)$ denote the Chow ring for Y. Then there is a proper morphism $f : Y' \longrightarrow Y$ such that $f^* : A(Y) \longrightarrow A(Y')$ is injective and $f^*(E)$ splits, i.e. it has a filtration $f^*(E) = E_0 \supset E_1 \supset \cdots, E_r = 0$ whose successive quotients are all Γ-line bundles.*

Proof: Let $\mathbb{P}(E)$ be the projective bundle associated to E. The variety $\mathbb{P}(E)$ is a Γ–variety with the Γ–action induced by the action of Γ on E. Let $p : \mathbb{P}(E) \longrightarrow Y$ denote the canonical projection. The line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $P(E)$ is indeed a Γ–line bundle. This follows because of the isomorphism $\mathbb{P}(\gamma^*(E)) \cong \mathbb{P}(E)$ for each $\gamma \in \Gamma$. We have an exact sequence $0 \longrightarrow \mathcal{K} \longrightarrow p^*(E) \longrightarrow \mathcal{O}_{\mathbb{P}(E)}(1) \longrightarrow 0$(see [4, Proposition 7.11]). Now we do the same process to $\mathcal{K}$ to obtain the required filtration for $p^*(E)$.

q.e.d

Definition 2.17. *Let L_* be a parabolic line bundle on X with a parabolic structure on D with a parabolic weight $\alpha = m/N$ a rational number. We define*

$$par_{c_1}(L_*) := c_1(L) + \alpha D.$$

Now let E_* be any parabolic torsion free sheaf on X. Then we define:

$$par_{c_1}(E_*) = c_1(E) + \sum_{i=1}^{l} \alpha_i r_i D$$

where r_i=rank$(F_i(E)/F_{i+1}(E))$. Here $F_i(E)/F_{i+1}(E)$ is supported on D and whose rank is coming from Euler characteristic of it on D.

The splitting principle for the Γ–sheaves can be used to define the higher parabolic Chern classes for parabolic sheaves. I.e. we can define the parabolic Chern classes for parabolic sheaves using the splitting principle for parabolic sheaves. Let $c_i(E_*)$ denote the ith parabolic Chern class of the parabolic sheaf. Then we have

$$c_i(W) = p^* c_i(E_*)$$

where $E_* = p_*^\Gamma(W)$(cf. [16, Page 1318]).

One has the following notion of parabolic Euler characteristic for parabolic bundles.

In the parabolic category we always fix an ample line bundle Θ_1 on X as a polarisation unless we state otherwise. For a parabolic sheaf E_* we write $E_*(n)$ for the sheaf $E_* \otimes n\Theta_1$.

Definition 2.18. *Let E_* be a parabolic coherent sheaf on X. Let the polynomial $P(c_1(E), c_2(E), \cdots, c_r(E))$ denotes the usual Euler characteristic for E, where $c_i(E)$ are the Chern classes of the underlying sheaf E. Then we define the parabolic Euler characteristic*

$$\chi_{par}(E_*) := P(c_1(E_*), c_2(E_*), \cdots, c_r(E_*)).$$

Remark 2.19. In fact for the case of surfaces, using Riemann-Roch theorem one can write $\chi(E) = \frac{c_1(E)(c_1(E)-K_X)}{2} - c_2(E) + r\chi(\mathcal{O}_X)$ where r is the rank of the sheaf E.

Remark 2.20. Let P be a polynomial in $\mathbb{Q}[z]$. Let E_* be a parabolic sheaf. Then we define $\chi_{par}(E(n)_*)$ as the parabolic Hilbert polynomial for the sheaf E_*. We say E_* is a parabolic sheaf with Parabolic Hilbert polynomial P if $\chi_{par}(E(n)_*) = P(n)$. We write $\chi(E_*)$ for the parabolic Euler characteristic of E_* from now on. Also we write $P_{E_*}(n)$ for parabolic Hilbert polynomial $\chi_{par}(E(n)_*)$ of E_*. We write $p_{E_*}(n)$ for the reduced parabolic Hilbert polynomial $\frac{\chi_{par}(E(n)_*)}{r}$.

Remark 2.21. Let L be any line bundle on X with trivial parabolic structure. Let $c_i(E_*)$ denote the ith parabolic Chern class of E_*. Then we have

$$parc_1(E_* \otimes L) = c_1(E \otimes L) + (\sum_{i=1}^{i=l} r_i \alpha_i)D = c_1(E_*) + rc_1(L)$$

and

$$\begin{aligned} parc_2(E_* \otimes L) &= c_2(E \otimes L) + \sum_{i=1}^{i=l} r_i \alpha_i (c_1(E \otimes L).D) \\ &- \sum_{i=1}^{i=l} \alpha_i (deg(F_i \otimes L) - deg(F_{i+1} \otimes L)) \\ &+ \frac{1}{2}\{(\sum_{i=1}^{i=l} r_i \alpha_i) \cdot (\sum_{j=1}^{j=l} r_j \alpha_j) - (\sum_{i=1}^{i=l} r_i \alpha_i^2)\} D^2 \\ &= c_2(E_*) + (r-1)c_1(L)c_1(E_*) + \frac{r(r-1)}{2} c_1(L)^2 \end{aligned}$$

Our definition of parabolic Hilbert polynomial of E_* for the case of $dim(X) = 2$ can be written as

$$p_{E_*}(n) = \frac{n^2\Theta_1^2}{2} + (\frac{c_1(\mathcal{E}_*) \cdot \Theta_1}{r} - \frac{K_X \Theta_1}{2})n + \frac{c_1(\mathcal{E}_*)^2 - 2c_2(\mathcal{E}_*) - c_1(\mathcal{E}_*) \cdot K_X}{2r} + \chi(\mathcal{O}_X) \tag{2.3}$$

and for an effective divisor D on X

$$\frac{\chi(E_*(D))(n)}{r} = \frac{\chi(E_*)(n)}{r} + nD \cdot \theta_1 + \frac{D^2}{2} + \frac{c_1(E_*)D}{r} - \frac{DK_X}{2}. \tag{2.4}$$

Definition 2.22. *A parabolic torsion free sheaf E_* on a smooth projective algebraic surface X is parabolic semistable if for all parabolic subsheaves F_* of E_*, we have the following inequality for sufficiently large N and $n \geq N$:*

$$(\frac{\chi(F_*(-D)(n))}{r(\mathcal{F})} + \frac{c_1(\mathcal{F}_*) \cdot D}{2r(\mathcal{F})}) \leq (\frac{\chi(E_*(-D)(n))}{r(\mathcal{E})} + \frac{c_1(\mathcal{E}_*) \cdot D}{2r(\mathcal{E})})$$

Lemma 2.23. *A parabolic torsion free sheaf E_* is parabolic χ-semistable(stable) on X if and only if the corresponding Γ–sheaf W is (Γ, χ)-semistable(stable) on Y.*

Proof: We recall that we have fixed the polarisations Θ and Θ_1 for Y and X respectively. Let W be a Γ–coherent sheaf of Y. Then we have $c_i(W) = p^*(c_i(E_*))$, $K_Y = p^*(K_X) \otimes p^*(\mathcal{O}_X(D))$. We recall the following equation for the normalised Hilbert polynomial of W.

$$\frac{\chi(W(n))}{r} := \frac{n^2\Theta^2}{2} + n(\frac{c_1(W)\Theta}{r} - \frac{K_Y \cdot \Theta}{2}) + \frac{c_1(W)^2 - 2c_2(W) - c_1(W)\cdot K_Y}{2r} + \chi(\mathcal{O}_Y)$$

Let W_1 be any Γ–subsheaf of W so that the quotient is torsion free. And let F_* is the corresponding parabolic subsheaf of E_*.

Then we also have

$$\frac{\chi(\mathcal{E}_*(-D)(n))}{r} - \frac{\chi(\mathcal{F}_*(-D)(n))}{r_1} = \frac{\chi(\mathcal{E}_*(n))}{r} - \frac{\chi(\mathcal{F}_*(n))}{r_1} - (\frac{c_1(E_*)D}{r} - \frac{c_1(F_*)D}{r_1})$$

We rewrite the normalised Hilbert polynomial of W as follows:

$$\begin{aligned}
\frac{\chi(W(n))}{r} &= |\Gamma|[\frac{n^2\Theta_1^2}{2} + n(\frac{c_1(E_*)\Theta_1}{r} - \frac{K_X\Theta_1 + D\Theta_1}{2}) \\
&+ \frac{c_1(\mathcal{E}_*)^2 - 2c_2(\mathcal{E}_*) - c_1(\mathcal{E}_*)(K_X + D)}{2r}] + \chi(\mathcal{O}_Y) \\
&= |\Gamma|[\frac{\chi(E_*(n))}{r} - n\frac{D\cdot\Theta_1}{2} - \frac{c_1(\mathcal{E}_*)\cdot D}{2r} - \chi(\mathcal{O}_X)] + \chi(\mathcal{O}_Y)
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{\chi(W(n))}{r} - \frac{\chi(W_1(n))}{r_1} \\
= \; & |\Gamma|[\frac{\chi(E_*(n))}{r} - \frac{\chi(F_*(n))}{r_1} - (\frac{c_1(\mathcal{E}_*)\cdot D}{2r} - \frac{c_1(\mathcal{F}_*)\cdot D}{2r_1})] \\
= \; & |\Gamma|[\frac{\chi(E_*(-D)(n))}{r} - \frac{\chi(F_*(-D)(n))}{r_1} + (\frac{c_1(\mathcal{E}_*)\cdot D}{2r} - \frac{c_1(\mathcal{F}_*)\cdot D}{2r_1})] \\
= \; & |\Gamma|[(\frac{\chi(E_*(-D)(n))}{r} + \frac{c_1(\mathcal{E}_*)\cdot D}{2r}) - (\frac{\chi(F_*(-D)(n))}{r_1} + \frac{c_1(\mathcal{F}_*)\cdot D}{2r_1})]
\end{aligned}$$

So we conclude that the parabolic sheaf E_* is parabolic χ-semistable(stable) if and only if the corresponding Γ–sheaf W is (Γ, χ)-semistable(stable).

q.e.d

Remark 2.24. On higher dimensional varieties we do not know how to obtain the relation between the Hilbert polynomials as it involves Todd classes and pull backs of Chern classes.

Definition 2.25. *Let E_* be a parabolic χ-semistable sheaf on X with reduced parabolic Hilbert polynomial $p_{E_*}(n)$. Then there is a filtration*

$$E_1 \subset E_2 \subset E_3 \subset \cdots \subset E_n = E_*$$

of parabolic subsheaves E_i of E such that $E_1, E_2/E_1, \cdots E_n/E_{n-1}$ are parabolic χ-stable sheaves with $p_{(E_i/E_{i-1})_*}(n) = p_{E_*}(n)$. The sheaf

$$E_1 \oplus E_2/E_1 \oplus \cdots E_n/E_{n-1}$$

is denoted by $gr(E_*)$ and it is called the associated graded object of E_* associated to a Jordan-Hölder filtration. The sheaf $gr(E_*)$ is unique. We say two parabolic χ–semistable sheaves E_* and F_* are S-equivalent if $gr(E_*) \cong gr(F_*)$.

2.3. **Moduli functors for parabolic bundles.** Let X be a smooth projective algebraic surface over $\mathbb{C}$. Let D be a normal crossing divisor on X. let Θ_1 be an ample line bundle on X.

Suppose S is a scheme of finite type over $\mathbb{C}$ and let $p : X \times S \longrightarrow S$ the projection.

Definition 2.26. *A coherent torsion free sheaf $\mathcal{E}$ on $X \times S$ over S is said to have a parabolic structure if there is a filtration*

$$\mathcal{E} = F_1(\mathcal{E}) \supset F_2(\mathcal{E}) \supset \cdots \supset F_l(\mathcal{E}) \supset \mathcal{E} \otimes p^*\mathcal{O}(-D)$$

of $\mathcal{E}$ on the relative divisor $D \times S$ such that the sheaves $F_i(\mathcal{E})$ are flat over S and α_* the system of weights on the filtration F_*. In particular for each $s \in S$ the sheaf $\mathcal{E}_{*s}$ is a parabolic torsion free sheaf on X with the parabolic structure given by F_{*s} and α_*.

Definition 2.27. *We say a parabolic torsion free sheaf $\mathcal{E}_*$ on $X \times S$ over S with the parabolic Hilbert polynomial P is parabolic χ-semistable if it is a flat family of torsion free sheaves over S, and for each $s \in S$, the sheaf $\mathcal{E}_{s*}$ is a parabolic χ-semistable sheaf on X with parabolic Hilbert polynomial P.*

Let $Sch/\mathbb{C}$ denotes the category of schemes over $\mathbb{C}$. We fix a polynomial P in $\mathbb{Q}[z]$. We fix the following topological invariants: We fix a quasi parabolic structure F_*, the weights α_*, the degree of $F_i(E)$ on D denoted as l_i, the rank of F_i/F_{i+1} written as r_i. Let $\mathbf{s}_* := (\mathbf{r}, \mathbf{l}, \alpha)$ denote these invariants. In the words of [10] this is equivalent to fixing $(\chi(E)(m), \chi(E/F_i)(m), \alpha_*)$. We say a parabolic sheaf E_* with parabolic Hilbert polynomial P if $P(E_*)(n) = P(n)$ and the sheaf E_* has the parabolic structure given by the tuple $\mathbf{s}_* = (\mathbf{r}, \mathbf{l}, \alpha)$. We remark that P and $\mathbf{s}_*$ does not fix the underlying Chern classes of E_*.

Definition 2.28. *Let*

$$Par_M(T) := \{E_* | E_* \text{ is parabolic } \chi\text{-semistable on } X \times T \text{ over } T \text{ with } P(E_{*t}) = P\ \}/\sim$$

1. *For each $t \in T$, E_{*t} is a parabolic semistable sheaf with parabolic Hilbert polynomial P.*
2. *$E_* \sim F_*$ if and only if there are filtrations of $0 = E^k \subset E^{k-1} \cdots \subset E^0$ of E_* and $0 = F^k \subset F^{k-1} \cdots \subset F^0$ of F_* with the F^i, E^i are T-flat such that for each $t \in T$, their restrictions to X_t give the Jordan-Hölder filtrations of E_{*t} and F_{*t} respectively. In particular, $gr(E_{*t}) \cong gr(F_{*t})$. For a line bundle $\mathcal{L}$ on S, we assume that $gr(E_*) \cong gr(F_*) \otimes \mathcal{L}$.*

For a morphism $g : T' \longrightarrow T$, the pullback g^* defines a map $Par_M(T) \longrightarrow Par_M(T')$. Then Par_M is a contravariant functor. By using the categorical equivalence 2.14 we will prove that the functor Par_M has a coarse moduli scheme $M^{par}(P)$ which co-represents the S-equivalence classes of parabolic χ-semistable sheaves on X of the parabolic Hilbert polynomial P and parabolic datum $\mathbf{s}_*$.

In fact we show that the parabolic moduli functor is the same as Γ–moduli functor on a suitable Kawamata covering Y of X(see Section 4).

3. **Grassmannian varieties**

Let V and W be vector spaces. Let $Grass(V \otimes W, a) := G$ denote the Grassmannian of quotients of dimension a of $V \otimes W$. There is a canonical $SL(V)$ action on G. One knows that there is an embedding of the Grassmannian G into the projective space $\mathbb{P}^N$ where $N = \binom{dim(V\otimes W)}{a} - 1$. This embedding is the Plücker embedding which is given by $V \otimes W \longrightarrow U \longrightarrow 0$ to $\wedge^a(V \otimes W) \longrightarrow \wedge^a(U) \longrightarrow 0$. This embedding defines a very ample line bundle $\mathcal{L}$ on G. In fact this is a canonical $SL(V)$-invariant projective embedding of G into the projective space $\mathbb{P}(\wedge^a(V \otimes W))$ given by the ample invertible sheaf $\mathcal{L}$.

Suppose that V has a Γ-action and W has a trivial Γ-action. Now the group Γ acts on the Grassmannian of quotients of dimension a of $V \otimes W$. The fixed points for the Γ-action are the quotients which are Γ-quotients of $V \otimes W$. Let $Grass_\Gamma(V \otimes W, a)$ denote the Γ–fixed points in G. This is a closed subscheme of G. Let $Aut_\Gamma(V)$ denote the Γ-invariant automorphisms of V. We see that the subgroup $H := SL(V) \cap Aut_\Gamma(V)$ acts on $Grass_\Gamma(V \otimes W, a)$. We restrict the line bundle $\mathcal{L}$ to this closed subscheme. Let $\mathcal{L}'$ denote the restricted line bundle. Then it is H–linearised. We also see that H is a direct product of linear groups(see proof of Proposition 3.1).

The following proposition characterises all the semistable(stable) points in the scheme $Grass_\Gamma(V \otimes W, a)$.

Proposition 3.1. *A point $\alpha : V \otimes W \longrightarrow L \longrightarrow 0$ in $Grass_\Gamma(V \otimes W, a)$ is semistable for the action of H and a line bundle $\mathcal{L}'$ if and only if for all non-zero proper Γ–subspaces $V' \subset V$ we have $Image(V' \otimes W)$ in L is not zero and*

$$\frac{dim(V')}{dim(Image(V' \otimes W))} \leq \frac{dim(V)}{dim(L)} \tag{3.1}$$

(properly stable if the inequality is strict)

Proof:

Since V is a Γ–vector space we can find Γ–invariant irreducible subspaces V_i of V so that $V = \oplus_{i\in I} V_i$ where we may have $V_i \cong V_j$. Let $m := |I|$. In fact any Γ–subspace V' of V can be written as $V' = \oplus_{j\in J} V_i$ where $J \subset I$. If one writes $V = \oplus m_i V_i$ for distinct i, we see that $Aut_\Gamma V = \prod GL(m_i)$. Let k_i denote the dimension of V_i. Let $\lambda(t)$ be a 1 PS(one parameter subgroup) of $Aut_\Gamma V$. Now by choosing a basis of V, the 1 PS $\lambda(t)$ can be written as

$$\lambda(t) = \begin{bmatrix} t^{r_1}.I_1 & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & t^{r_m}.I_m \end{bmatrix} \tag{3.2}$$

where I_i is the identity matrix of rank k_i and $\sum k_i r_i$ need not be zero. Indeed any 1 PS $\lambda_i(t)$ of $Aut_\Gamma(V_i \otimes \mathbb{C}^{m_i})$ where $\mathbb{C}^{m_i}$ is the vector space of dimension m_i counting the repetition of V_i can be written as

$$\lambda_i(t) = \begin{bmatrix} t^{r_1}.I_i & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & t^{r_{m_i}}.I_i \end{bmatrix} \tag{3.3}$$

where I_i is the identity matrix of rank k_i.

Therefore any 1 PS $\lambda(t)$ of H can be written as (3.2) with one more condition $\sum_I k_i r_i = 0$.

Since any Γ–automorphism of V_i is a scalar, each number r_i appears k_i times for each i. Since $\alpha : V \otimes W \longrightarrow L$ is a Γ–map, the image $\alpha(U \otimes W)$ is a Γ–subspace of L for a Γ-subspace $U \subset V$.

In order to study the semistability criterion we use the numerical criterion (cf. [**?**, Page 87]). We compute the number $\mu(L, \lambda)$ for our map α and 1 PS $\lambda(t)$.

We recall the following computation of $\mu(L, \lambda)$ given in [**?**, Page 158]. Let V and W be vector spaces of dimension n and m respectively. Let $\lambda(t)$ be a 1 PS of $SL(V)$. Let r be the dimension of the quotient L of $V \otimes W$. By choosing a basis of V, $\lambda(t)$ can be written as

$$\lambda(t) = \begin{bmatrix} t^{r_1} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & t^{r_n} \end{bmatrix} \tag{3.4}$$

Then

$$\mu(L, \lambda) = -rr_n + \sum_{j=1}^{n-i} (r_{j+1} - r_j) dim(\alpha(U_{jm}))$$

where U_{jm} is generated by $W_j \otimes W$ for any subspace W_j of V. We have $\mu(L, \lambda) > 0 (\geq 0)$ if and only if it is true for the extreme cases

$$r_1 = r_2 = \cdots = r_p = p - n$$

and

$$r_{p+1} = r_{p+2} = r_{p+3} \cdots = r_n = p$$

with $1 \leq p \leq n - 1$.

We note that in our case α is a Γ–map and $\lambda(t)$ is a 1 PS of H. We use the fact that V has a Γ–irreducible decomposition and any Γ–automorphism of V_i is a scalar multiplication. Let $\{e_{11}, e_{12}, e_{13}, \cdots, e_{1k_1}, e_{21}, \cdots, e_{2k_2} \cdots, e_{m1}, \cdots, e_{mk_m}\}$ be a basis of the Γ–vector space V. Then

$\mu(L, \lambda) > 0$ for all $r_{11} \leq r_{12} \leq \cdots \leq r_{mk_m}$ if and only if

$$-rp + n dim(\alpha(U_{pm})) > 0$$

for $1 \leq p \leq m - 1$ where $U_{pm} = W_p \otimes W$ for a Γ–subspace W_p of V.

So we have

$$\frac{dim(\alpha(W_p \otimes W))}{dim(L)} > \frac{dim(W_p)}{dim(V)}.$$

The same is true for the case of semistability. Thus we have the proposition.

q.e.d

3.1. **Quot schemes.** Let Y be a smooth projective variety over $\mathbb{C}$. Fix a very ample line bundle Θ on Y. Let $\mathcal{W}$ be a coherent sheaf on Y. From now on we write $\mathcal{F}(m) := \mathcal{F} \otimes m\Theta$ for each $m \in \mathbb{Z}$.

Grothendieck constructed the Quot scheme $Quot(\mathcal{W}, P)$ parametrising quotients

$$\mathcal{W} \longrightarrow \mathcal{F} \longrightarrow 0$$

with Hilbert polynomial P. The Quot scheme $Quot(\mathcal{W}, P)$ represents the functor

$$\mathbf{Quot}(\mathcal{W}, P) : (Sch/\mathbb{C})^o \longrightarrow Sets$$

defined as for any scheme S, the S-valued points of $\mathbf{Quot}(\mathcal{W}, P)(S)$ are the isomorphism classes of quotients on $X \times S$,

$$p^*\mathcal{W} \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{F}$ is flat over S and $P(\mathcal{F}, n) = P(n)$.

Grothendieck proved that $Quot(\mathcal{W}, P)$ is projective over $\mathbb{C}$. In fact there is an M such that for any $m \geq M$ we get a map:

$$\psi_m : Quot(\mathcal{W}, P) \longrightarrow Grass(H^0(X, \mathcal{W}(m)), P(m)).$$

He also proved that for sufficiently large M, the map ψ_m is a closed embedding.

Let $\tilde{\rho} : \mathcal{O}_{Quot(\mathcal{W},P)} \otimes \mathcal{W} \longrightarrow \tilde{\mathcal{F}}$ be the universal quotient module parametrised by the scheme $Quot(\mathcal{W}, P)$. Then the line bundle $\mathcal{L}_m := det(p_*(\tilde{\mathcal{F}} \otimes q^* m\Theta)($ where $p : Quot(\mathcal{W}, P) \times X \longrightarrow Quot(\mathcal{W}, P)$ and $q : Quot(\mathcal{W}, P) \times X \longrightarrow X$ are the projections) is a very ample line bundle on the scheme $Quot(\mathcal{W}, P)$. This line bundle $\mathcal{L}_m$ is the pullback of the canonical invertible sheaf $\mathcal{L}$ on the Grassmannian given by the plücker embedding.

Suppose that V is a Γ–vector space. The group $SL(V)$ acts on $Quot(V \otimes \mathcal{W}, P)$ induced by the action of $SL(V)$ on V. The group $SL(V)$ also acts on the line bundle $\mathcal{L}_m$. Let $Quot_\Gamma(V \otimes \mathcal{W}, P)$ denote the Γ–fixed points of the Quot scheme $Quot(V \otimes \mathcal{W}, P)$. Note that this scheme is closed. This scheme parametrises Γ–quotients $V \otimes \mathcal{W} \longrightarrow \mathcal{F} \longrightarrow 0$ with Hilbert polynomial P. The group $H := SL(V) \cap Aut_\Gamma(V)$ acts on this closed subscheme $Quot_\Gamma(V \otimes \mathcal{W}, P)$. Let $\mathcal{L}'_m$ be the pullback line bundle $\mathcal{L}_m | Quot_\Gamma(V \otimes \mathcal{W}, P)$. It is easy to see that the line bundle $\mathcal{L}'_m$ is H–linearised.

We study this scheme in the next section in more detail.

We can now describe the properly stable and semistable points on the scheme $Quot_\Gamma(V \otimes \mathcal{W}, P)$ with respect to the H–group action and the line bundle $\mathcal{L}'_m$.

Lemma 3.2. *There exists an M such that for $m \geq M$ the following holds. Suppose $V \otimes \mathcal{W} \longrightarrow \mathcal{F} \longrightarrow 0$ is a point in $Quot_\Gamma(V \otimes \mathcal{W}, P)$. For any subspace $V_1 \subset V$ let $\mathcal{G}$ denote the subsheaf of $\mathcal{F}$ generated by $V_1 \otimes \mathcal{W}$. Suppose that $P(\mathcal{G}, m) > 0$ and*

$$dim(V_1)/P(\mathcal{G}, m) \leq dim(V)/P(m)$$

for all non-zero proper Γ–subspaces V_1. Then the point is H–semistable with respect to $\mathcal{L}'_m$ and the group action of H. The strict inequality holds for stable points.

Proof: For large m the Γ–fixed points of the Quot scheme is embedded in the Γ–fixed points of the Grassmannian scheme $Grass_\Gamma(V \otimes W, P(m))$ where $W := H^0(\mathcal{W}(m))$. For all points in $Quot_\Gamma(V \otimes \mathcal{W}, P)$ and all Γ–subspaces U, the Γ–sheaves $\mathcal{G}$ run over a bounded family. Let $\mathcal{K}$ denote the kernel of the Γ–exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow U \otimes \mathcal{W} \longrightarrow \mathcal{G} \longrightarrow 0. \tag{3.5}$$

We can choose M large so that for $m \geq M$, $h^0(\mathcal{G}(m)) = P(\mathcal{G}(m))$ and $h^1(\mathcal{K}(m)) = 0$ for all such $\mathcal{G}$ and $\mathcal{K}$. Note that

$$Im(U \otimes W) \subset H^0(\mathcal{G}(m)) \subset H^0(\mathcal{F}(m)).$$

The above exact sequence induces

$$0 \longrightarrow \mathcal{K}(m) \longrightarrow U \otimes \mathcal{W}(m) \longrightarrow \mathcal{G}(m) \longrightarrow 0$$

and hence the following long exact sequence of cohomology, which is

$$U \otimes W \longrightarrow H^0(\mathcal{G}(m)) \longrightarrow H^1(\mathcal{K}(m)) \longrightarrow \cdots .$$

Since $H^1(\mathcal{K}(m)) = 0$, we get $dim(Im(H \otimes W)) = P(\mathcal{G}(m))$. Now using Proposition (3.1) in the previous section we get the result.

q.e.d

4. **Moduli of Γ–semi stable sheaves**

We fix Y and Θ as in the previous section. We note that the family of Γ–semistable sheaves on Y with fixed Hilbert polynomial P is bounded. This follows because the Γ–semistable sheaves in particular are semistable sheaves on Y. Fix a large number N so that any Γ–sheaf with Hilbert polynomial P is globally generated and $h^0(\mathcal{F}(N)) = P(N)$. Let $\mathcal{E}$ be a Γ–sheaf on Y so that the underlying vector bundle is trivial and $h^0(\mathcal{E}) = P(N)$. Let $\mathcal{W} = \mathcal{O}_Y(-N)$ and $V = H^0(\mathcal{E})$. We note that V is a Γ–module of rank $P(N)$.

We define $Quot_\Gamma(V \otimes \mathcal{W}, P)$ functorially as follows:

For a scheme S the set of S-valued points in $Quot_\Gamma(V \otimes \mathcal{W}, P)$ may be described as the set of pairs

$$\{(\mathcal{F}, \alpha) | P(\mathcal{F}_s) = P, \alpha : V \otimes \mathcal{O}_S \longrightarrow H^0(Y \times S/S, \mathcal{E}(N))\}$$

where $\mathcal{F}$ is a Γ–coherent sheaf on $Y \times S$, flat over S with Hilbert polynomial of $\mathcal{F}_s$ is P and α is a Γ–morphism so that the sections in the image of α generates the Γ–module $\mathcal{F}(N)$.

Let $Q_1 \subset Quot_\Gamma(V \otimes \mathcal{W}, P)$ denote the open set where $\mathcal{E}$ has pure dimension d and it is $(\Gamma - \chi)$-semistable. Since the set of χ-semistable sheaves with Hilbert polynomial P is bounded we can choose a large N, so that every $(\Gamma - \chi)$-semistable sheaf with Hilbert polynomial P appears as a Γ–quotient sheaf corresponding to a point in Q_1.

Let $R^\Gamma \subset Q_1$ be a scheme consisting of sheaves $q \in Q_1$ so that the linear map α is a Γ–isomorphism.

We also fix a large M so that for each $m \geq M$ there exists an embedding $\psi_m : Quot_\Gamma(V \otimes \mathcal{W}, P) \longrightarrow Grass_\Gamma(V \otimes H^0(Y, \mathcal{W}(m)), P(m))$ corresponding to the line bundle $\mathcal{L}'_m = \mathcal{L}_m|_{Quot_\Gamma(V \otimes \mathcal{W}, P)}$. The group H acts on $Quot_\Gamma(V \otimes \mathcal{W}, P)$ and on $\mathcal{L}'_m$. It is clear from the definition that R^Γ is invariant under this action of H.

Remark 4.1. Let $Quot_\Gamma(V \otimes \mathcal{W}, P, d)$ denote the closure in $Quot_\Gamma(V \otimes \mathcal{W}, P)$ of the set of points such that the quotient sheaf $\mathcal{E}$ is a pure d dimensional sheaf on Y. Then we have $R^\Gamma \subset Quot_\Gamma(V \otimes \mathcal{W}, P, d)$.

We recall the following facts from [15]. We rewrite some of the statements for Γ–sheaves which follow easily from the proofs given in [15].

Lemma 4.2. [15, Lemma 1.16] *Let $\mathcal{F}$ be the Γ–sheaf represented by a point of $Quot_\Gamma(V \otimes \mathcal{W}, P)$. Suppose there exists an integer M such that for $m \geq M$, $\mathcal{F}$ is semistable with respect to the line bundle $\mathcal{L}'_m$ and the action of H, then the following property holds. For any non-zero Γ–subspace $V_1 \subset V$, let $\mathcal{G} \subset \mathcal{F}$ be the* *subsheaf generated by* $V_1 \otimes \mathcal{W}$. *Then the* $rank(\mathcal{G}) > 0$ *and* $dim(V_1)/rank(\mathcal{G}) \leq dim(V)/rank(\mathcal{F})$.

Remark 4.3. [15, remark of Lemma 1.16] In the situation of the above lemma, suppose $\mathcal{F} \longrightarrow \mathcal{K} \longrightarrow 0$ is a quotient sheaf of $\mathcal{F}$. Let $V_1 \subset V$ be the kernel of the map $V \longrightarrow H^0(Hom(\mathcal{W}, \mathcal{K}))$, and let J be the image. We have $dim(V_1) = dim(V) - dim(J)$. If $\mathcal{G}$ is the subsheaf generated by $H \otimes \mathcal{W}$, then $\mathcal{G}$ maps to zero in $\mathcal{K}$, so $r(\mathcal{G}) \leq r(\mathcal{F}) - r(\mathcal{K})$. Thus the conclusion of the lemma implies that

$$\frac{dim(J)}{r(\mathcal{K})} \geq \frac{dim(V)}{r(\mathcal{F})}.$$

By a similar argument for $\mathcal{K} = \mathcal{F}$, we can conclude that the map $V \longrightarrow H^0(Hom(\mathcal{W}, \mathcal{F}))$ is injective.

Remark 4.4. The above remark states that, for a Γ-sheaf $\mathcal{E} \in Quot_\Gamma(V \otimes \mathcal{W}, P, d)$ to be semistable, it is necessary that the induced homomorphism $V \longrightarrow H^0(\mathcal{E}(m))$ is injective and that no $(d-1)$-dimensional subsheaf $\mathcal{F}'$ of $\mathcal{E}$ has a global section.

Lemma 4.5. [15, lemma 1.17]

If $\mathcal{E}$ *is the sheaf represented by a point of* $Quot_\Gamma(V \otimes \mathcal{W}, P, d)$, *let* $\mathcal{I}$ *be the coherent subsheaf of sections supported in dimension* $\leq d-1$. *Then there is a* Γ*–sheaf* $\mathcal{E}'$ *of pure dimension* d, *with Hilbert polynomial* P, *and an inclusion* $0 \longrightarrow \mathcal{E}/\mathcal{I} \longrightarrow \mathcal{E}'$.

Proof: Note that $\mathcal{I} = T_{d-1}(\mathcal{E})$ is the torsion subsheaf of $\mathcal{E}$ which appears in the torsion filtration of $\mathcal{E}$. So the sheaf $\mathcal{I}$ is a Γ–sheaf. We can find a curve C, a point $0 \in C$ and a morphism $C \longrightarrow Quot_\Gamma(V \otimes \mathcal{W}, P)$ such that $\{0\}$ goes to the point corresponding to $\mathcal{E}$ and all other points to pure d dimension sheaves. Therefore we have a C-flat family E of d-dimensional Γ sheaves on X such that $E_0 \cong \mathcal{E}$ and E_c is pure for all $c \in C - \{0\}$. By following the proof of Simpson we obtain the Γ–injection $0 \longrightarrow \mathcal{E}/\mathcal{I} \longrightarrow \mathcal{E}'$ where $\mathcal{E}'$ is a pure d- dimensional Γ–sheaf on Y.

Lemma 4.6. [15, Lemma 1.18] *There exists an* N_0 *such that for all* $N \geq N_0$, *the following is true. Suppose* $\mathcal{E}$ *is a* χ*-semistable sheaf on* Y, *with Hilbert polynomial* P, *then for all subsheaves* $\mathcal{F} \subset \mathcal{E}$, *we have*

$$h^0(\mathcal{F}(N))/r(\mathcal{F}) \leq P(N)/r(\mathcal{E}) \tag{4.1}$$

and if the equality holds then

$$P(\mathcal{F}, m)/r(\mathcal{F}) = P(m)/r(\mathcal{E}) \qquad \forall m. \tag{4.2}$$

We now prove that the set of all (Γ, χ) semistable(stable) sheaves is indeed the semistable(stable) points in the scheme $Quot_\Gamma(V \otimes \mathcal{W}, P, d)$ for the action of H and the embedding given by the line bundle $\mathcal{L}'_m$. This theorem is very crucial in a way that it characterizes all the semistable points in the projective scheme $Quot_\Gamma(V \otimes \mathcal{W}, P, d)$.

Theorem 4.7. *Let* P *be a polynomial of degree* d *with rational coefficients. There exist* N *and* M *such that for* $m \geq M$, *the following is true. A point* $\mathcal{E}$ *in* $Quot_\Gamma(V \otimes \mathcal{W}, P, d)$ *is semistable (stable) for the action of* H *with respect to the embedding determined by* m, *if and only if the quotient* $\mathcal{E}$ *is* $(\Gamma - \chi)$ *semistable(*$(\Gamma - \chi)$*-stable) coherent sheaf of pure dimension* d *and* $V \cong H^0(\mathcal{E}(N))$ *is a* Γ*–isomorphism.*

Proof: Suppose we have a point $(\mathcal{E},\alpha)$ in $Quot_\Gamma(V\otimes\mathcal{W},P,d)$ such that $\mathcal{E}$ is a $(\Gamma-\chi)$-semistable sheaf and the map α is a Γ–isomorphism. By Lemma 3.2, it is enough to prove that we may choose $M(N)$ which is dependent on N such that for $m\geq M$ and for any subsheaf $\mathcal{F}\subset\mathcal{E}$ generated by Γ–sections of $\mathcal{E}(N)$ we have

$$h^0(\mathcal{F}(N))/P(\mathcal{F},m)\leq P(N)/P(m) \tag{4.3}$$

and the inequality is strict for Γ–stable points. We may choose N_0 such that for any $N\geq N_0$ and any point in $Quot_\Gamma(V\otimes\mathcal{W},P)$ representing a semistable sheaf $\mathcal{E}$, the conclusion of the above Lemma 4.6 holds.

The above Lemma 4.6 in particular holds for $(\Gamma-\chi)$-semistable sheaves as they are usual χ-semistable sheaves. So the inequality in Lemma 4.6 is true for Γ–subsheaves also.

Suppose $\mathcal{E}$ is (Γ,χ)-stable. Then for large m,

$$P(\mathcal{F},m)/r(\mathcal{F})<P(\mathcal{E},m)/r(\mathcal{E})$$

where $\mathcal{F}$ are Γ–invariant subsheaves of $\mathcal{E}$. Once the N is fixed the set of all Γ–subsheaves $\mathcal{F}$ generated by Γ–sections of $\mathcal{E}(N)$ is bounded, so the set of polynomials $P(\mathcal{F},m)$ is finite. These polynomials all have leading term $r(\mathcal{F})m^d$. Consider the leading coefficients of the polynomials which appear in the inequality (4.3). The terms $r(\mathcal{E})h^0(\mathcal{F}(N))$ and $P(N)r(\mathcal{F})$ are the leading coefficients of the polynomials $h^0(\mathcal{F}(N))P(m)$ and $P(N)P(\mathcal{F},m)$ respectively. We note that for the Γ–subsheaves $\mathcal{F}$ of $\mathcal{E}$,

$$h^0(\mathcal{F}(N))/r(\mathcal{F})<P(N)/r(\mathcal{E}).$$

If not, then since $\mathcal{E}$ is (Γ,χ) semistable (which is underlying semistable) we have

$$P(\mathcal{F},m)/r(\mathcal{F})=P(\mathcal{E},m)/r(\mathcal{E})$$

for all m, by using Lemma 4.6 which contradicts the fact that, the sheaf $\mathcal{E}$ is (Γ,χ)-stable. So we can choose large $M(N)$ so that we have

$$h^0(\mathcal{F}(N))/P(\mathcal{F},m)\leq P(N)/P(m),$$

by using the inequality $h^0(\mathcal{F}(N))/r(\mathcal{F})<P(N)/r(\mathcal{E})$ on the leading coefficients. So we have the result for the stable case.

For the semistable case for a given $\mathcal{F}$ subsheaf we have the inequality,

$$h^0(\mathcal{F}(N))/r(\mathcal{F})<P(N)/r(\mathcal{E})$$

or the equality

$$h^0(\mathcal{F}(N))/r(\mathcal{F})=P(N)/r(\mathcal{E})\quad\text{and}\quad P(\mathcal{F},m)/r(\mathcal{F})=P(\mathcal{E},m)/r(\mathcal{E})$$

for all m by using Lemma 4.6. In the first case we prove by the same way as above. For the second case we substitute the value of $P(\mathcal{F},m)$ and $h^0(\mathcal{F}(N))/r(\mathcal{F})$ on the left hand side of the inequality (4.3) we get the required result proving the semistable case.

Now we prove the converse. Let $N\geq N_0$. Suppose $V\otimes\mathcal{W}\longrightarrow\mathcal{F}\longrightarrow 0$ be a point in $Quot_\Gamma(V\otimes\mathcal{W},P,d)$ which is H-semistable with respect to the embedding ψ_m for $m\geq M$. Let $T:=T_{d-1}(\mathcal{F})$ be the torsion sheaf of $\mathcal{F}$ and let $\mathcal{E}$ be the sheaf of pure dimension d given by Lemma 4.5. So the Hilbert polynomial of $\mathcal{E}$ is P. We show that all sheaves $\mathcal{E}$ obtained this way remain in a bounded family independent of N. In particular one can choose large N_0 so that for $N\geq N_0$, $h^0(\mathcal{E}(N))=P(N)$ and $\mathcal{E}(N)$ is globally generated. Now we apply Remark 4.3 to the sheaf $\mathcal{F}/T$, we get that $h^0(\mathcal{F}/T)(N) \geq P(N)$. But this sheaf is a subsheaf of $\mathcal{E}$, thus we conclude that $\mathcal{F}/T = \mathcal{F}$. Since the Hilbert polynomials of $\mathcal{E}$ and $\mathcal{F}$ are the same we have $T = 0$. Thus $\mathcal{E} = \mathcal{F}$ is a pure sheaf of dimension d and remains in a bounded family independent of N.

Now by again using Lemma 4.2 for the sheaf $\mathcal{F}$ which is H-semistable we see that the map $V \longrightarrow H^0(\mathcal{F}(N))$ is Γ–injective (This action induced by the Γ–map $V \otimes \mathcal{W} \longrightarrow \mathcal{F} \longrightarrow 0$), by dimension count we conclude that it is a Γ–isomorphism.

It remains to prove that the sheaf $\mathcal{F}$ is (Γ, χ)-semistable. If $\mathcal{F}$ is not (Γ, χ)-semistable we have a quotient $\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$ such that

$$P(\mathcal{G}, k)/r(\mathcal{G}) < P(\mathcal{F}, k)/r(\mathcal{F})$$

for large k. Again we can assume that the sheaves $\mathcal{G}$ remain in a bounded family, and hence for large N_0, so that for $N \geq N_0$,

$$P(\mathcal{G}, N)/r(\mathcal{G}) < P(\mathcal{F}, N)/r(\mathcal{F})$$

and also $h^0(\mathcal{G}(N)) = P(\mathcal{G}, N)$. But this contradicts Lemma 4.2 because $\mathcal{F}$ is a H-semistable point. Thus the sheaf $\mathcal{F}$ is (Γ, χ)-semistable.

Finally we assume that the sheaf $\mathcal{F}$ is not (Γ, χ)-stable but it is an H-stable point. Since $\mathcal{F}$ is in particular an H-semistable point we see that it is a (Γ, χ)-semistable sheaf. Then by using Lemma 4.6 we conclude that there is a Γ–subsheaf $\mathcal{E}$ the subsheaf of $\mathcal{F}$ with the condition $P(\mathcal{E})/r(\mathcal{E}) = P(\mathcal{F})/r(\mathcal{F})$. Let $U = H^0(\mathcal{E}(m))$ and $W = H^0(\mathcal{W}(m)) = H^0(\mathcal{O}_X(m - N))$ where $\mathcal{W} = \mathcal{O}_X(-N)$ and we let $V_1 = H^0(\mathcal{F}(N)) \subset V$. Again the sheaves $\mathcal{F}$ of this type remain in a bounded family, so we can assume that there are large m and N so that

$$Im(V_1 \otimes W) = H^0(\mathcal{F}(m)) \subset U$$

and

$$dim(V_1)/h^0(\mathcal{F}(m)) = P(m)/h^0(\mathcal{E}(m)).$$

Now the criterion of Proposition 3.1 says that $\mathcal{F}$ maps under ψ_m to a point in the Grassmannian which is not properly stable under the group H. Hence this sheaf $\mathcal{F}$ is not properly H-stable in the Quot scheme contradicting the assumption on $\mathcal{F}$.

q.e.d

Remark 4.8. The scheme R^Γ is equal to the set of H–semistable points of $Quot_\Gamma(V \otimes \mathcal{W}, P, d)$ under the action of H. The open set $R^{\Gamma^s} \subset R^\Gamma$ parametrising (Γ, χ)-stable sheaves is equal to the set of properly stable points under H.

Remark 4.9. Let $Quot_\tau(V \otimes \mathcal{W}, P, d)$ be the set of all Γ–torsion free sheaves of fixed local type τ in the projective scheme $Quot_\Gamma(V \otimes \mathcal{W}, P, d)$. By the rigidity of the representations of the finite groups, we see that this scheme is a closed subscheme. Let R^τ denote the open subscheme of $Quot_\tau(V \otimes \mathcal{W}, P, d)$ parametrising the (Γ, χ)-semistable sheaves of local type τ.

Lemma 4.10. *The closures of the orbits of two points $\mathcal{W}_1$ and $\mathcal{W}_2$ in R^τ intersect if and only if $gr_\Gamma(\mathcal{W}_1) \cong gr_\Gamma(\mathcal{W}_2)$. In particular they are of same local type τ. The orbit of $\mathcal{W}$ is closed if and only if $\mathcal{W}$ is Γ-polystable.*

Proof: Note that for a given extension $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ of Γ–sheaves we can find a family of extensions E_t of E'' by E', parametrised by $t \in \mathbb{A}^1$, such that for each $t \neq 0$ the extension is isomorphic to the given one, and for $t = 0$

the extension is trivial. Suppose the Jordan-Hölder filtration for E is given by $0 = E_0 \subset E_1 \subset E_2 = E$. Then we find that the orbit corresponding to $gr_\Gamma(E)$ is in the closure of the orbit corresponding to E. Now we use induction on the length of the filtration. Therefore, if $gr_\Gamma(\mathcal{W}_1) = gr_\Gamma(\mathcal{W}_2)$, then the closures of the orbits of $\mathcal{W}_1$ and $\mathcal{W}_2$ intersect.

Conversely, we have to prove that the orbit corresponding to $gr_\Gamma(\mathcal{W})$ is closed. Suppose $\mathcal{W}$ is a (Γ, χ)-semistable sheaf such that $gr_\Gamma(\mathcal{W}) \cong \mathcal{W}$. Suppose T is a curve and $t_0 \in T$ is a closed point, and suppose that $\mathcal{E}$ is a (Γ, χ)-semistable sheaf on $Y \times T$ over T such that $\mathcal{E}_t \cong \mathcal{W}$ for $t \neq t_0$. If $\mathcal{W}_i$ is a Γ–stable component of $\mathcal{W}$ then there are at least as many Γ–maps from $\mathcal{W}_i$ to $\mathcal{E}_{t_0}$ as to $\mathcal{W}$. Since $\mathcal{E}_{t_0}$ is semistable, this implies that $\mathcal{E}_{t_0}$ is a direct sum of copies of $\mathcal{W}_i$ with the same multiplicities as $\mathcal{W}$, so $\mathcal{E}_{t_0} = \mathcal{W}$.

q.e.d

We recall the construction of the Γ–frame bundle associated to a Γ–vector bundle as in [1].

4.0.1. *Γ–frame bundle.* Let S be a scheme of finite type with a *trivial* Γ–action. Let F be a Γ–locally free $\mathcal{O}_S$ module of rank r and assume that each fibre F_s is a Γ–module and the Γ–module structures on all points are same. Let W be a finite dimensional vector space of dimension r which is a Γ–module isomorphic to the Γ–module F_s for $s \in S$. Denote by $\mathcal{O}_S(W)$ the trivial rank r sheaf modelled by W. With this added structure, we have a canonical group namely, $H_1 = Aut_\Gamma(W) \subset GL(W)$, which acts on $\mathcal{O}_S(W)$ by Γ–automorphisms.

Let $\mathbb{H}om_\Gamma(\mathcal{O}_S(W), F) := Spec(S^*(\mathcal{H}om_\Gamma(\mathcal{O}_S(W), F)))^* \to S$ be the geometric Γ–vector bundle that parametrises all Γ–homomorphisms from $\mathcal{O}_S(W)$ to F. Let $\Phi(F) := \mathbb{I}som_\Gamma(\mathcal{O}_S(W), F) \subset \mathbb{H}om_\Gamma(\mathcal{O}_S(W), F)$ be the open subscheme which parametrises all Γ–isomorphisms and let $\pi : \Phi(F) \to S$ denote the canonical projection.

Then we observe that the group H_1 acts on $\Phi(F)$ by composition and π is a principal bundle with structure group H_1. Indeed, the Γ–structure on F gives a natural reduction of structure group of the frame bundle associated to F (which by the usual construction is a principal $GL(W)$-bundle).

This bundle $\pi : \Phi(F) \to S$ is called the Γ–frame bundle associated to F. It follows from the construction that there is a Γ–isomorphism $\phi : \mathcal{O}^r_{\Phi(F)} \longrightarrow \pi^* F$, called as the universal trivialisation of F.

We now define the Γ–moduli functor as follows:

Let Y be a smooth projective variety over $\mathbb{C}$. Let Θ be an ample line bundle on Y. Fix a polynomial $P \in \mathbb{Q}[z]$. We define a Γ–moduli functor

$$\mathcal{M}^\Gamma : (Sch/\mathbb{C})^0 \longrightarrow Sets$$

as follows. If S is a scheme over $\mathbb{C}$, let $\mathcal{M}^\Gamma(S)$ be the set of isomorphism classes of S-flat families of Γ–semistable sheaves on Y with Hilbert polynomial P. If $f : S' \longrightarrow S$ is a morphism of schemes, let $\mathcal{M}^\Gamma(f) : \mathcal{M}^\Gamma(S) \longrightarrow \mathcal{M}^\Gamma(S')$ be given by $[F] \longrightarrow [f^* F]$.

We consider the quotient $\mathcal{M}^\Gamma / \sim$ again denoted by $\mathcal{M}^\Gamma$ where the equivalence relation means $F \sim F'$ for $F, F' \in \mathcal{M}^\Gamma(S)$ if and only if $F \cong F' \otimes p^*(L)$ for some line bundle $L \in Pic(S)$. If we consider the Γ–stable sheaves only we have the open subfunctor $(M^{\Gamma})^s$.

Remark 4.11. If we restrict to the case of Γ–semistable sheaves of local type τ, we get the Γ–moduli functor $\mathcal{M}^{\tau}$ as the moduli functor for Γ-semistable sheaves of local type τ.

Theorem 4.12. *The Γ–moduli functor $\mathcal{M}^{\Gamma}$ is corepresented by a scheme M^{Γ}. And the scheme M^{τ} corepresents the Γ–moduli functor $\mathcal{M}^{\tau}$.*

Proof: Let F be an S-flat family of Γ–semistable sheaves with Hilbert polynomial P parametrised by S. Then $V_F := p_*(F \otimes q^*(N\Theta))$ is a Γ–locally free sheaf on S of rank $P(N)$. We have a canonical Γ–surjection $\phi_F : p^*(V_F) \otimes q^*(\mathcal{O}_Y(-N)) \longrightarrow F$. Let $R(F)$ denote the Γ–frame bundle associated to V_F with the natural projection $\pi : R(F) \longrightarrow S$. Composing ϕ_F with the universal trivialisation $f : \mathcal{O}^{P(N)} \longrightarrow \pi^*(F)$ of V_F on $R(F)$ we obtain a canonically defined quotient

$$q_F : \mathcal{O}_{R(F)} \otimes V \otimes \mathcal{O}_Y(-N) \longrightarrow \pi_Y^*(F)$$

on $R(F) \times Y$. This Γ–quotient gives rise to a classifying morphism

$$\Phi_F : R(F) \longrightarrow Quot_{\Gamma}(V \otimes \mathcal{O}_Y(-N), P).$$

Since $R(F)$ is a Γ–frame bundle this is an $H_1 := Aut_{\Gamma}(P(N), \mathbb{C})$-principal bundle, and the morphism Φ_F is H_1-equivariant. Therefore we have $\Phi_F(R(F)) \subset Quot_{\Gamma}(V \otimes \mathcal{O}_Y(-N), P)$. In this way one can prove that there is a natural transformation between $\mathcal{M}^{\Gamma}$ and R^{Γ}/H. The same follows for M^{τ}.

We recall the following theorem:

Theorem 4.13. [4, Theorem 4.2.10] *Let G be a reductive group acting on a projective scheme X with a G-linearised ample line bundle L. Then there is a projective scheme Y and a morphism $\pi : X^{ss}(L) \longrightarrow Y$ such that π is a universal good quotient for the G-action. Moreover there is an open subset $Y^s \subset Y$ such that $X^s(L) = \pi^{-1}(Y^s)$ and such that $\pi : X^s(L) \longrightarrow Y^s$ is a universal geometric quotient. Finally, there is a positive integer m and a very ample line bundle M on Y such that $L^{\otimes m}|_{X^{ss}(L)} \cong \pi^{-1}(M)$.*

We apply the above theorem to the scheme R^{Γ} with a group action H and with a H linearised line bundle L'_m coming from the embedding ψ_m. So we have the following theorem:

Theorem 4.14. *Let Y be a smooth projective algebraic variety over $\mathbb{C}$. Let P be a polynomial in $\mathbb{Q}[z]$. Let V be a Γ–vector space of dimension $P(N)$. Let R^{Γ} the scheme which parametrises (Γ, χ)-semistable pure d-dimensional coherent sheaves on Y with the fixed Hilbert polynomial P. Then there is good quotient $M^{\Gamma} := R^{\Gamma}//H$ which is a projective scheme. The points of M^{Γ} represent the equivalence classes of (Γ, χ)-semistable pure d-dimensional sheaves on Y under the relation that $\mathcal{W}_1 \sim \mathcal{W}_2$ if $gr_{\Gamma}(\mathcal{W}_1) = gr_{\Gamma}(\mathcal{W}_2)$. Furthermore, there is an open set $(M^{\Gamma})^s$ which represents all the isomorphism classes of (Γ, χ)-stable sheaves on Y.*

Proof:

The existence of the good quotient M^{Γ} is coming from the above theorem. The projectiveness of the scheme M^{Γ} follows from the Seshadri's results [12] since $R^{\Gamma} \subset$

$Quot_{\Gamma}(V \otimes \mathcal{W}, P, d)$ is the set of all semistable points of the projective scheme $Quot_{\Gamma}(V \otimes \mathcal{W}, P, d)$.

In order to check the equivalence relation, it is enough to check that for the sheaves $\mathcal{W}_1, \mathcal{W}_2$, the closures of the corresponding orbits in the scheme R^{Γ} intersect if and only if $gr_{\Gamma}(\mathcal{W}_1) = gr_{\Gamma}(\mathcal{W}_2)$. This comes from Lemma 4.10.

Lemma 4.15. *Let R^{τ} denote the subset of R^{Γ} consisting of all $q \in R^{\Gamma}$ such that $\mathcal{F}_q$ is locally of a fixed type τ. Then there is a good quotient $M^{\tau} := R^{\tau}//H$. This is a moduli space of Γ–semistable sheaves of local type τ. Again we have the equivalence relation $\sim$ that $\mathcal{F}_1 \sim \mathcal{F}_2$ if and only if $gr_{\Gamma}(\mathcal{W}_1) = gr_{\Gamma}(\mathcal{W}_2)$. Furthermore, there is an open set $(M^{\tau})^s$ which represent all the isomorphic classes of (Γ, χ)-stable sheaves on Y of local type τ.*

Proof: We see that any (Γ, χ)-semistable sheaf of local type τ is a semistable point in $Quot_{\Gamma}(V \otimes \mathcal{W}, P, d)$. The group H acts on this scheme R^{τ}. The scheme R^{τ} is embedded in R^{Γ} as a closed subscheme. Therefore the good quotient $M^{\tau} = R^{\tau}//H$ exists. If $\mathcal{E}_1 \sim \mathcal{E}_2$ then they are of the same local type. This implies that the moduli space M^{τ} corepresents equivalence classes of (Γ, χ)-semistable pure d-dimensional sheaves of local type τ on Y under the relation that $\mathcal{W}_1 \sim \mathcal{W}_2$ if $gr_{\Gamma}(\mathcal{W}_1) = gr_{\Gamma}(\mathcal{W}_2)$.

Lemma 4.16. *Let X be a smooth projective algebraic surface over $\mathbb{C}$. Let Θ_1 be a very ample line bundle on X. Let D be a normal crossing divisor on X. Let Y be a Kawamata covering of X as above. Suppose E_* is a S-flat parabolic torsion free sheaf on X. Then there is a S-flat Γ–torsion free sheaf W on Y such that $p_*^{\Gamma}(W) = E_*$.*

Lemma 4.17. *Parabolic moduli functor Par_M defined in 2.28 is corepresented by a scheme M^{τ}.*

Proof: Let E_* be an S-flat family of parabolic torsion free sheaves on X with parabolic Hilbert polynomial P and parabolic datum $\mathbf{s}_*$. In particular this is a parabolic sheaf on $S \times X$. This implies that there is W a Γ–sheaf on $S \times Y$. I.e. W is a S-flat family of Γ–torsion free sheaves on Y with Hilbert polynomial P'(induced by P) and of local type τ. We see that the functors Par_M and $\mathcal{M}^{\tau}$ are equivalent. This follows because we have a morphism $Par_M(S) \longrightarrow \mathcal{M}^{\tau}(S)$ for each S. In fact this is an isomorphism(Again follows from Seshadri-Biswas correspondence). Therefore corepesenting the functor Par_M is equivalent to corepresenting the functor $\mathcal{M}^{\tau}$

q.e.d

Thus we have the following theorem on parabolic bundles on X.

For notations see the Section 2.3.

Theorem 4.18. *Let X be a smooth projective algebraic surface over $\mathbb{C}$. Let Θ_1 be a ample divisor on X. Let D be a normal crossing divisor on X. Let P be a polynomial in $\mathbb{Q}[z]$. We fix $\mathbf{s}_*$ a parabolic datum. Then there is a moduli space $M^{par}(P)$ of parabolic χ-semistable bundles E_* on X with parabolic Hilbert polynomial P_* and parabolic datum $\mathbf{s}_*$.*

5. Determinant line bundles on the parabolic moduli space

Let Y be a smooth projective variety of dimension n equipped with a very ample line bundle Θ. Let $K(Y)$ and $K^0(Y)$ denote the Grothendieck groups of coherent sheaves and locally free sheaves on Y respectively. Then $K^0(Y)$ is a commutative ring with $1 = [\mathcal{O}_Y]$, with the multiplication given by the tensor product of locally free sheaves. Since Y is smooth and projective, we have $K(Y) = K^0(Y)$. This ring $K(Y)$ is equipped with a quadratic form $q : (u, v) \longrightarrow \chi(Y, u \otimes v)$. This form is calculated in terms of the rank and the Chern classes of u. For example, if Y is a smooth projective surface, and if $u \in K(Y)$ is of rank r, and the Euler characteristic χ, we have

$$q(u \otimes u) = 2r\chi + c_1^2 - r^2\chi(\mathcal{O}_Y)$$

We say $u, v \in K(Y)$ are numerically equivalent, $u \equiv v$ if $u - v$ is in the radical of the quadratic form q. We work with the quotient $K(Y)_{num} := K(Y)/\equiv$.

If $\mathcal{F}$ is a flat family of coherent sheaves on Y parametrised by a scheme S, then $\mathcal{F}$ defines an element $[\mathcal{F}] \in K^0(S \times Y)$, the Grothendieck group of $S \times Y$ generated by locally free sheaves. Let p, q are the projections from $Y \times S$ to S and Y respectively. We define a homomorphism from the Grothendieck group of coherent sheaves on Y:

$$\lambda_{\mathcal{F}} : K(Y) \longrightarrow Pic(S).$$

as follows: For $u \in K(Y)$, $\lambda_{\mathcal{F}}(u) = det(p_!(\mathcal{F} \cdot q^*(u))$, where $\mathcal{F} \cdot q^*(u)$ is the product in $K(S \times Y)$ and $p_! : K^0(S \times Y) \to K^0(S)$ associates to each class u the class $\sum_i (-1)^i R^i p_*(u)$.

We observe that this $\lambda_{\mathcal{F}}$ has a collection of functorial properties(see [4, Page 179]):

Lemma 5.1.

(1) *If $0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0$ is a short exact sequence of S-flat families of coherent sheaves then $\lambda_{E_2} \cong \lambda_{E_1} \otimes \lambda_{E_3}$.*

(2) *If F is an S-flat family and $f : S' \longrightarrow S$ a morphism then for any $u \in K(Y)$ one has $\lambda_{f_Y^* F}(u) = f^* \lambda_F(u)$.*

(3) *If G is an algebraic group, S a scheme with a G-action and E a G-linearised S-flat family of coherent sheaves on Y, then λ_E factors through the group $Pic^G(S)$ of isomorphism classes of G-linearised line bundles on S.*

(4) *Let $\mathcal{E}$ be an S-flat family of coherent sheaves of class $c \in K_{num}(Y)$ and let $\mathcal{N}$ be a locally free $\mathcal{O}_S$ sheaf. Then $\lambda_{\mathcal{F} \otimes p^* \mathcal{N}}(u) = \lambda_{\mathcal{F}}(u)^{r(\mathcal{N})} \otimes det(\mathcal{N})^{\chi(c \otimes u)}$*

For any class $c \in K(Y)_{num}$, we write $c(m) := c \cdot [m\Theta]$ and denote by $P(c)$ the associated Hilbert polynomial $P(c, m) = \chi(c(m))$.

Let $\mathcal{F}$ be an S-flat family of Γ–coherent sheaves with Hilbert polynomial $P(c)$. The points $s \in S$ such that $\mathcal{F}_s$ is of class c form an open and closed subscheme of S. Therefore $M^\Gamma(P)$ decomposes into finitely many open and closed subschemes of $M^\Gamma(c_i)$ where $P(c_i) = P$.

Let $S \subset K(Y)$ be any subset, let $S^\perp \subset K(Y)$ be the subset of all the elements orthogonal to S with respect to the quadratic form q.

Definition 5.2. *Let $c \in K_{num}(Y)$ and Θ a very ample divisor on Y with $\theta = [\mathcal{O}_\Theta] \in K(Y)$. Then we define $K_c := c^\perp$ and $K_{c,\Theta} := c^\perp \cap \{1, \theta, \theta^2, \cdots \theta^n\}^{\perp\perp}$*

We recall the following descent lemma from [4, Page 87].

Theorem 5.3. *Let $\pi : X \longrightarrow Y$ be a good quotient. Let $\mathcal{F}$ be a G-linearised locally free sheaf on X. A necessary and sufficient condition for $\mathcal{F}$ to descend is that for* *any point $x \in X$ in the closed orbit xG, the stabiliser G_x of x acts trivially on the fibre $\mathcal{F}_x$.*

We have the following theorem which says that the condition $u \in K_{c,\Theta}$ is a sufficient condition to get a well defined determinant line bundle on $M^{\Gamma}(c)$ by means of u.

Theorem 5.4. *Let $c \in K_{num}(Y)$ be a class. Then there are group homomorphisms $\lambda^s : K_c \longrightarrow Pic(((M^{\Gamma})^s)(c))$ and $\lambda : K_{c,\Theta} \longrightarrow Pic((M^{\Gamma})(c))$ with the following properties:*

(1) *λ and λ^s commute with the inclusion $K_{c,\Theta} \subset K_c$ and the restriction $Pic((M^{\Gamma})(c)) \longrightarrow Pic(((M^{\Gamma})^s)(c))$.*
(2) *Let $\mathcal{E}$ be a S-flat Γ–semistable torsion free sheaf of class c on Y, and if $\phi_{\mathcal{E}} : S \longrightarrow M^{\Gamma}(c)$ is the classifying morphism, then λ and $\lambda_{\mathcal{E}} : K(Y) \longrightarrow Pic(S)$ commute with the inclusion $K_{c,\Theta} \subset K(Y)$ and the homomorphism $\phi^*_{\mathcal{E}} : Pic(M^{\Gamma}(c)) \longrightarrow Pic(S)$.*
(3) *If $\mathcal{E}$ is a S-flat family of Γ–stable sheaves of class c on Y, then λ^s and $\lambda_{\mathcal{E}} : K(Y) \longrightarrow Pic(S)$ commute with the inclusion $K_c \subset K(Y)$ and the homomorphism $\phi^*_{\mathcal{E}} : Pic((M^{\Gamma}(c)))^s \longrightarrow Pic(S)$.*

Remark 5.5. If we consider the moduli space $M^{\tau}(c)$ of Γ–semistable sheaves of class c and of local type τ we see that the above theorem holds true in this case also.

Proof: Let $R^{\Gamma}(c) \subset Quot_{\Gamma}(V \otimes \mathcal{W}, P, d)$ denote the open subscheme of Γ–quotients $V \otimes \mathcal{O}_Y(-N) \longrightarrow \mathcal{F} \longrightarrow 0$ with $\mathcal{F}$ a Γ–semistable sheaf of class c and a Γ–isomorphism $V \longrightarrow H^0(\mathcal{F}(N))$. Then there is an universal quotient $\mathcal{O}_{R^{\Gamma}(c)} \otimes V \otimes \mathcal{O}_Y(-N) \longrightarrow \tilde{\mathcal{F}}$. If we choose large M and $m \gg 0$, $R^{\Gamma}(c)$ is the set of semistable points in the closure $\overline{R^{\Gamma}(c)}$ with respect to the action of H and the H-linearised line bundle $\lambda_{\tilde{\mathcal{F}}}([m\Theta])$. Moreover $M^{\Gamma}(c) = R^{\Gamma}(c)//H$.

Let $u \in K(Y)_{num}$ be any class. We consider the line bundle $L := \lambda_{\tilde{\mathcal{F}}}(u)$ on $R^{\Gamma}(c)$. We note that the line bundle L inherits a H-linearisation from $\tilde{\mathcal{F}}$. We claim that L descends to a line bundle on $M^{\Gamma}(c)$ or $(M^{\Gamma})^s(c)$ if $u \in K_{c,\Theta}$ or $u \in K_c$ respectively.

By using Theorem 5.3 it is enough to check that for any point $q : V \otimes \mathcal{W} \longrightarrow \mathcal{F} \longrightarrow 0$ in a closed H-orbit, the action of the stabiliser group H_q on the fibre L_q is trivial. Note that the H-orbit in the scheme $R^{\Gamma}(c)$ is closed if and only if it is a Γ–polystable sheaf.

Let $\mathcal{F}$ be a Γ–polystable sheaf in the H-closed orbit. Then $\mathcal{F} \cong \oplus_i(\mathcal{F}_i \otimes W_i)$ with distinct Γ–stable sheaves $\mathcal{F}_i$ and vector spaces W_i. Consider the corresponding morphism $\{q\} \longrightarrow R^{\Gamma}(c)$. Then we see that $L_q = (\lambda_{\tilde{\mathcal{F}}}(u))_q$ as a pullback of the determinant line bundle L to the point $\{q\}$. We rewrite the fibre L_q as follows. Since W_i is a vector space we can think of this as a trivial bundle on $\{q\}$. So on $X \times \{q\}$, we have $R^{\bullet}p_*(\mathcal{F}_i \otimes p^*(W_i) \otimes q^*(u)) = R^{\bullet}p_*(\mathcal{F}_i \otimes q^*(u)) \otimes W_i$. Then by using Lemma 5.1 we have $(\lambda_{\tilde{\mathcal{F}}_i \otimes p^*(W_i)}(u)) = (\lambda_{\tilde{\mathcal{F}}_i}(u))^{dim(W_i)} \otimes det(W_i)^{\chi([\mathcal{F}_i]\otimes u)}$ where $[\mathcal{F}_i]$ is a class of $\mathcal{F}_i$ in $K(Y)$. This implies that

$$L_{\mathcal{F}} \cong \otimes_i (det(H^*([\mathcal{F}_i] \cdot u))^{dim(W_i)} \otimes (det(W_i)^{\chi([\mathcal{F}_i]\cdot u)}).$$

We see that $H_{\mathcal{F}} = Aut_{\Gamma}(\mathcal{F}) \cong \Pi GL(W_i)$, and an element $(A_1, A_2, \cdots, A_l)$ with $A_i \in GL(W_i)$ acts on the fibre $L_q = (\lambda_{\tilde{\mathcal{F}}}(u))_q$ via multiplication with the number $\Pi_i det(A_i)^{\chi(u\cdot[\mathcal{F}_i])}$.

Let $c_i = [\mathcal{F}_i]$ and r, r_i are the ranks of $\mathcal{F}$, $\mathcal{F}_i$ respectively. We have for all m:

$$r_i\chi(c\cdot[m\Theta]) = r_iP(\mathcal{F}(m)) = rP(\mathcal{F}_i(m)) = r\chi(c_i\cdot[m\Theta]).$$

This is equivalent to $\chi((rc_i - r_ic)\cdot\theta^m) = 0$ for all m, so we have $(rc_i - r_ic) \in \{1,\theta,\theta^2,\cdots\theta^n\}^{\perp}$.

Suppose $\mathcal{F}$ is Γ–stable. Then $Aut(\mathcal{F}) \cong \mathbb{C}^*$. So the action of $A \in \mathbb{C}^*$ on the fibre $L_{\mathcal{F}}$ is the multiplication by $A^{\chi(u.c)}$. Therefore for $u \in K_c$ the multiplication is trivial because $\chi(u.c) = 0$.

For $\mathcal{F}$ is not Γ–stable, assume $u \in K_{c,\Theta}$. Then we have $\chi((rc_i - r_ic)\cdot u) = 0$ for $(rc_i - r_ic) \in \{1,\theta,\theta^2,\cdots,\theta^n\}^{\perp}$. This implies that $\chi(u.c_i) = \frac{r_i}{r}\chi(u.c) = 0$. Therefore $\Pi_i det(A_i)^{\chi(u\cdot[\mathcal{F}_i])} = 1$.

Hence we have proved that the line bundle $\lambda_{\tilde{\mathcal{F}}}(u)$ descends to the line bundle $\lambda^s(u)$ on $(M^\Gamma)^s(c)$ for $u \in K_c$ or to the line bundle $\lambda(u)$ on $M^\Gamma(c)$ for $u \in K_{c,\Theta}$.

Now we check the commutative properties of the diagrams:

The commutativity of the following diagram follows by the definitions of the maps.

(5.1)
$$\begin{array}{ccc} K_{c,\Theta} & \xrightarrow[\text{inclusion}]{} & K_c \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\lambda^s} \\ Pic(M^\Gamma(c)) & \xrightarrow[\text{restriction}]{} & Pic(M^\Gamma)^s(c) \end{array}$$

Suppose $\mathcal{E}$ is an S-flat family of Γ–semistable sheaves of class c. Then the sheaf $p_*(\mathcal{E}\otimes q^*l\Theta)$ is a locally free sheaf of rank $P(l)$ for large l. This sheaf is in fact given by $V := H^0(S, p_*(\mathcal{E}\otimes q^*l\Theta))$. This sheaf is a Γ–sheaf on S(the action of Γ on S is trivial). Let $\pi : \tilde{S} = \mathbb{I}som(\mathcal{O}_S(V), p_*(\mathcal{E}\otimes q^*l\Theta)) \longrightarrow S$ be the Γ–frame bundle associated to the above sheaf. Let $\tilde{\phi_{\mathcal{E}}} : \tilde{S} \longrightarrow R^\Gamma(c)$ be the classifying morphism for the Γ–quotient $V\otimes\mathcal{O}_{\tilde{S}\times Y} \longrightarrow \pi^*(\mathcal{E}) \longrightarrow 0$ which is the composition of the Γ–isomorphism $\phi : \mathcal{O}_{\tilde{S}}^{P(l)} \longrightarrow \pi^*\mathcal{E}$ and the evaluation map. Since H acts on $\tilde{S}$ the map ϕ is an H-equivariant morphism. So the map $\tilde{\phi_{\mathcal{E}}}$ is an H-equivariant morphism, and we have $\pi'\circ\tilde{\phi_{\mathcal{E}}} = \phi_{\mathcal{E}}\circ\pi$, where $\phi_{\mathcal{E}} : S \longrightarrow M^\Gamma(c)$ is the classifying morphism for the family $\mathcal{E}$ and $\pi' : R^\Gamma(c) \longrightarrow M^\Gamma(c)$ is a good quotient. We also see that the map $\pi^* : Pic(S) \longrightarrow Pic^H(\tilde{S})$ is an injective map.

(5.2)
$$\begin{array}{ccc} \tilde{S} & \xrightarrow[\tilde{\phi_{\mathcal{E}}}]{} & R^\Gamma(c) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ S & \xrightarrow[\phi_{\mathcal{E}}]{} & M^\Gamma(c) \end{array}$$

From the above diagram we have the following sequence of H-equivariant isomorphisms.

$$\pi^*{\phi_{\mathcal{E}}}^*\lambda(u) = \tilde{\phi_{\mathcal{E}}}^*\pi'^*\lambda(u) = \tilde{\phi_{\mathcal{E}}}^*\lambda_{\tilde{F}}(u) = \lambda_{\tilde{\phi_{\mathcal{E}}}^*\tilde{F}}(u) = \lambda_{\pi^*\mathcal{E}}(u) = \pi^*\lambda_{\mathcal{E}}(u).$$

From the above equalities we have the following: For $u \in K_{c,\Theta}$, $\lambda_{\mathcal{E}}(u) = \phi_{\mathcal{E}}^*\lambda(u)$. So we have proved the second part. The third part follows easily from the above calculation and the second part. Thus we have the theorem.

q.e.d

We recall the following classes which are defined in [4]:

For Y a smooth projective algebraic variety of dimension n, fix a very ample divisor Θ on Y and a class $c \in K(Y)_{num}$ and a base point $x \in Y$. Let $\theta = [\mathcal{O}_\Theta] \in K(Y)$. Define for each $i \in \{0, 1, \cdots n\}$:

$$u_i(c) := -r \cdot \theta^i + \chi(c \cdot \theta^i) \cdot [\mathcal{O}_x].$$

We see that $\chi(c \otimes u_i(c)) = 0$. If $b \in \{1, \theta, \theta^2, \cdots \theta^n\}^\perp$, then $\chi(\theta^n \cdot b) = rank(b)deg(X) = 0$ implies that $rank(b) = 0$. We get $\chi(b \otimes u_i(c)) = 0$. Hence $u_i(c) \in K_{c,\Theta}$. Then by the above theorem we have the line bundles $\lambda(u_i(c))$ on $M^\Gamma(c)$ for each i. Let $\mathcal{L}_i$ denote the line bundle $\lambda(u_i(c))$ on $Pic(M^\Gamma(c))$. Then the restriction of these line bundles to the fibres $det^{-1}(\mathcal{Q})$ of the determinant map $det : M^\Gamma(c) \longrightarrow Pic(Y)$ is independent of the choice of x. So on $M^\Gamma(r, \mathcal{Q}, c_2)$ these line bundles are well defined (cf [4, page 183,184]).

Theorem 5.6. *Let $f_k : M^\Gamma(c) \longrightarrow M^\Gamma(c(k))$ be the isomorphism of the moduli spaces given by $[F] \longrightarrow [F \otimes k\Theta]$. For sufficiently large $k \gg 0$, the line bundle $\mathcal{L}_0$ is ample on $M^\Gamma(c(k))$ and in particular it is ample on $M^\Gamma(c)$.*

Proof: The proof follows easily from [4, Theorem 8.1.11].

q.e.d

From all the above discussion we have following conclusion;
Let Y be a smooth projective surface with an ample divisor Θ. Let c be class in $K(Y)_{num}$ with rank r and Chern classes c_1, c_2, and a line bundle $\mathcal{Q}$ with $c_1(\mathcal{Q}) = c_1$. Then the line bundle $\mathcal{L}_0 \otimes \mathcal{L}_1^m$ is ample on the moduli space $M^\Gamma(r, \mathcal{Q}, c_2)$ for sufficiently large $m >> 0$. In particular we have line bundles $\mathcal{L}_i$ on the moduli space $M^\tau(c)$ of Γ–torsion free sheaves of class c and of local type τ. And the line bundle $\mathcal{L}_0 \otimes \mathcal{L}_1^m$ is ample on $M^\tau(c)$.

In [1], it has been shown that the linear system $|\mathcal{L}_1^m|$ is base point free and we have constructed $M^{\Gamma,\mu ss}(r, \mathcal{Q}, c_2)$ the moduli space of (Γ, μ)-semistable sheaves on Y(Y is a surface).

For easy reference we recall the construction of the Donaldson-Uhlenbeck compactification.

Let Y be a smooth projective algebraic surface and let Θ be a very ample line bundle on Y. Let $\mathcal{W} = \mathcal{O}_Y(-N)$ and let $V = H^0(E)$ be a Γ–module where E is a Γ–sheaf with underlying vector bundle is trivial. Let P be any polynomial in $\mathbb{Q}[z]$. From now on we write $\mathcal{E} := V \otimes \mathcal{W}$. Let $R^{\Gamma,\mu}$ be an open subscheme of all (Γ, μ)-semistable quotients $[q : \mathcal{E} \longrightarrow \mathcal{V}]$ of $\mathcal{E}$ with fixed topological data $(r, \mathcal{Q}, c_2)$ given by the rank r, determinant $\mathcal{Q}$, second Chern class c_2 and such that q induces a Γ-isomorphism $V \longrightarrow H^0(\mathcal{V}(N))$. Note that the open subscheme $R^\Gamma(r, \mathcal{Q}, c_2)$ which parametrises all the Γ–quotients which are (Γ, χ)-semistable torsion free sheaves is embedded as a subscheme in $R^{\Gamma,\mu}$. Let $\mathcal{F}$ be the universal quotient for the scheme $R^{\Gamma,\mu}$. And let $\tilde{\mathcal{F}}$ be the corresponding universal quotient sheaf of $R^\Gamma(r, \mathcal{Q}, c_2)$.

The group H acts on the scheme $R^{\Gamma,\mu}$ by automorphisms. There is an H–linearised line bundle $\mathcal{M} := \lambda_{\mathcal{F}}(u_1(c))$ on $R^{\Gamma,\mu}$ where $u_1(c)$ as given before. Then we have the following lemma and corollary:

Lemma 5.7. [1, Lemma 3.7]

1. *If $s \in \mathcal{R}^{\Gamma,\mu}$ is a point such that for a general high degree Γ–invariant curve C, $\mathcal{F}_s \mid_C$ is semistable then there exists an integer $N > 0$ and an H–invariant section $\tilde{\sigma} \in H^0(\mathcal{R}^{\Gamma,\mu}, \mathcal{M}^N)^H$ such that $\tilde{\sigma}(s) \neq 0$.*
2. *If s_1 and s_2 are two points in $\mathcal{R}^{\Gamma,\mu}$ such that for a general high degree Γ–invariant curve C, $\mathcal{F}_{s_1} \mid_C$ and $\mathcal{F}_{s_2} \mid_C$ are both semistable but not S–equivalent or one of them is semistable but the other is not then there is a H–invariant section $\tilde{\sigma}$, in some tensor power of $\mathcal{M}$ which separates these two points (i.e $\tilde{\sigma}(s_1) = 0$ but $\tilde{\sigma}(s_2) \neq 0$).*

Corollary 5.8. *There exists an integer $\nu > 0$ such that the line bundle $\mathcal{M}^\nu$ on $\mathcal{R}^{\Gamma,\mu}$ is generated by H–invariant global sections.*

Remark 5.9. From the above lemma and the corollary we conclude that the line bundle $\mathcal{L}_1$ is base-point free on the moduli space $M^\Gamma(r, \mathcal{Q}, c_2)$. In particular it is base point free on the moduli space $M^\tau(r, \mathcal{Q}, c_2)$ of (Γ, χ)–semistable sheaves of local type τ.

Since $\mathcal{R}^{\Gamma,\mu}$ is a quasi-projective scheme and since $\mathcal{M}$ is H-*semi-ample*, there exists a finite dimensional vector space $A \subset A_\nu := H^0(\mathcal{R}^{\Gamma,\mu}, \mathcal{M}^\nu)^H$ that generates $\mathcal{M}^\nu$; of course, there is nothing canonical in the choice of A.

Let $\phi_A : \mathcal{R}^{\Gamma,\mu} \to \mathbb{P}(A)$ be the induced H-invariant morphism defined by the sections in A.

We denote by M_A the scheme theoretic image $\phi_A(\mathcal{R}^{\Gamma,\mu})$ with the canonical reduced scheme structure. We see that M_A is proper over $\mathbb{C}$. We see that the space $\bigoplus_{k\geq 0} H^0(R^{\Gamma,\mu}, M^{kS})^H$ is a finitely generated ring for sufficiently large number S.

We choose a large integer S . Let $M^{\Gamma,\mu}(r, \mathcal{Q}, c_2)$ be the projective scheme

$$\operatorname{Proj} \bigoplus_{k\geq 0} H^0(R^{\Gamma,\mu}, M^{kS})^H \tag{5.3}$$

and let

$$\phi : \mathcal{R}^{\Gamma,\mu}(r, \mathcal{Q}, c_2) \to M^{\Gamma,\mu}(r, \mathcal{Q}, c_2)$$

be the canonical induced morphism.

We note that there is an H–invariant morphism

$$\Phi : R^\Gamma(r, \mathcal{Q}, c_2) \longrightarrow \operatorname{Proj} \bigoplus_{k\geq 0} H^0(R^{\Gamma,\mu}, M^{kS})^H$$

which is the composition of the inclusion $R^\Gamma(r, \mathcal{Q}, c_2) \subset R^{\Gamma,\mu}(r, \mathcal{Q}, c_2)$ and the morphism ϕ. Then by the universal property of the categorical quotient map $\pi : R^\Gamma(r, \mathcal{Q}, c_2) \longrightarrow M^\Gamma(r, \mathcal{Q}, c_2)$ there is an unique morphism

$$\gamma : M^\Gamma(r, \mathcal{Q}, c_2) \longrightarrow M^{\Gamma,\mu}(r, \mathcal{Q}, c_2)$$

such that the following diagram commutes.

$$\begin{array}{ccc} R^\Gamma(r, \mathcal{Q}, c_2) & \xrightarrow{\Phi} & M^{\Gamma,\mu}(r, \mathcal{Q}, c_2) \\ {\scriptstyle\pi}\downarrow & \nearrow_{\gamma} & \\ M^\Gamma(r, \mathcal{Q}, c_2) & & \end{array}$$

We see that the line bundle $\mathcal{O}(1)$ on $M^{\Gamma,\mu}(r, \mathcal{Q}, c_2)$ pull backs to the line bundle $\lambda_{\tilde{\mathcal{F}}}(u_1(c))^S$ on $R^\Gamma(r, \mathcal{Q}, c_2)$ which in turn descends to the line bundle $\lambda(u_1(c)) = \mathcal{L}_1^S$

on $M^{\Gamma}(r, \mathcal{Q}, c_2)$. This morphism γ is the Gieseker-to-Uhlenbeck map for the Γ–category.

Proposition 5.10. *The moduli space $M^{\Gamma,\mu s}(r, \mathcal{Q}, c_2)$ of isomorphism classes of (Γ, μ)–stable locally free sheaves with fixed determinant $\mathcal{Q}$ on Y, is embedded in the moduli space $M^{\Gamma,\mu}(r, \mathcal{Q}, c_2)$.*

The closure of the moduli space $M^{\Gamma,\mu s}(r, \mathcal{Q}, c_2)$ in $M^{\Gamma,\mu}(r, \mathcal{Q}, c_2)$ is the desired Donaldson-Uhlenbeck compactification of (Γ, μ)-stable bundles.

From now on we restrict our attention to the category of Γ–sheaves of fixed topological local type τ. We rephrase the statements we have proved to this category. Then

Theorem 5.11. *Let $M^{\tau}(r, \mathcal{Q}, c_2)$ be the moduli space of (Γ, χ)-semistable sheaves of local type τ. And let $M^{\tau,\mu}(r, \mathcal{Q}, c_2)$ be the moduli space of (Γ, μ)-semistable sheaves of local type τ. Then there is a morphism γ called the Gieseker-Uhlenbeck map*

$$\gamma : M^{\tau}(r, \mathcal{Q}, c_2) \longrightarrow M^{\tau,\mu}(r, \mathcal{Q}, c_2)$$

such that $\gamma^*(\mathcal{O}(1)) = \mathcal{L}_1^S$ for some large S.

The Donaldson-Uhlenbeck compactification for parabolic bundles can be described as a stratified space in terms of parabolic polystable *bundles* with decreasing parabolic chern classes κ as follows:(For details see [1, Page 67])

$$\overline{M^{\boldsymbol{\alpha}}_{k,\mathbf{j},\mathbf{r}}(r, \mathcal{P}, \kappa)} \subset \coprod_{l \geq 0} M^{\boldsymbol{\alpha}-poly}_{k',\mathbf{j}',\mathbf{r}}(r, \mathcal{P}, \kappa - l) \times S^l(X). \tag{5.4}$$

where by $M^{\boldsymbol{\alpha}-poly}_{k,\mathbf{j},\mathbf{r}}(r, \mathcal{P}, \kappa)$, we mean the set of isomorphism classes of *polystable* parabolic bundles with parabolic datum given by $(\boldsymbol{\alpha}, \mathbf{l}, \mathbf{r}, \mathbf{j})$, fixed determinant $\mathcal{P}$ and with topological datum given by k and κ as mentioned above.

We have the following theorem in the parabolic category which is an immediate consequence of the previous theorem:

As we recall from the Section (2.3) we fix the parabolic datum $\mathbf{s}_*$.

Theorem 5.12. *Let X be a smooth projective algebraic surface over $\mathbb{C}$. Let Θ_1 be an ample divisor on X. Let D be a normal crossing divisor on X. Let P be a polynomial in $\mathbb{Q}[z]$. We fix $\mathbf{s}_*$ a parabolic datum. We also fix the tuple $\mathbf{c}_* = (\mathcal{P}, c_2, \mathbf{s}_*)$, where $\mathcal{P}$, the determinant of E_*, c_2 the second Chern class of the underlying sheaf of E_*. Let $M^{par}(\mathbf{c}_*)$ be the moduli space of χ-semistable parabolic bundles E_* of type $\mathbf{c}_*$ on X, and let $M^{par,\mu}(\mathbf{c}_*)$ be the moduli space of μ-semistable parabolic bundles E_* of type $\mathbf{c}_*$. Then there is a morphism γ called the Gieseker-to-Uhlenbeck map for the parabolic bundles:*

$$\gamma : M^{par}(\mathbf{c}_*) \longrightarrow M^{par,\mu}(\mathbf{c}_*).$$

6. Concluding remarks

- The moduli space we have constructed in this method is intrinsic at the level of smooth projective algebraic surfaces.
- Even though categorically Γ-sheaves and parabolic sheaves are the same, we could not prove the **Γ-semi-stable sheaf $\leftrightarrow$ parabolic semi-stable sheaf** under this correspondence for the reason that we could not obtain

the Lemma 2.23 for higher dimensions. In fact it is not even clear how to write the parabolic Chern character in higher dimensions for the parabolic sheaves (see. [5])

- The map will be very much helpful in the computations of wall crossing formula for the case of parabolic sheaves in the surface level.
- In our forthcoming paper we show that the morphism γ is strictly semismall.

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