4

RELATIVISTIC COVARIANCE AND KINEMATICS

4.1 REVIEW OF LORENTZ TRANSFORMATIONS

The special theory of relativity is based on two postulates:

1. The laws of nature are the same in two frames of reference in uniform relative motion with no rotation.
2. The speed of light is c in all such frames.

Let us consider two frames K and K', as shown in Fig. 4.1, with a relative uniform velocity v along the x axis. The origins are assumed to coincide at $t=0$. If a pulse of light is emitted at the origin at $t=0$, each observer will see an expanding sphere centered on his own origin. This is a consequence of postulate 2 and is inconsistent with classical concepts, which would have the sphere always centered on a point at rest in the "ether." The reconciliation of this result requires us to view both space and time as quantities peculiar to each observer and not universal. Therefore, we have the equations of the expanding sphere in each frame

$$x^2+y^2+z^2-c^2t^2=0, \qquad x'^2+y'^2+z'^2-c^2t'^2=0, \tag{4.1}$$

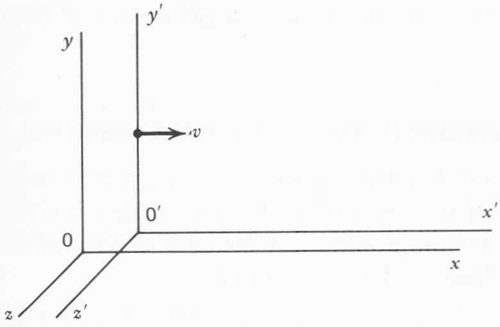


Figure 4.1 Two inertial frames with a relative velocity v along the x axis.

where t' does not equal t, as in Newtonian physics. The actual relations between $x\ y\ z\ t$ and $x'\ y'\ z'\ t'$ can be deduced by fairly elementary means if some further postulates (homogeneity and isotropy of space) are introduced. The result is called the *Lorentz transformation*:

$$x' = \gamma(x - vt) \tag{4.2a}$$

$$y' = y \tag{4.2b}$$

$$z' = z \tag{4.2c}$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right), \tag{4.2d}$$

where

$$\gamma \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{4.2e}$$

The inverse of this transformation is easily found:

$$x = \gamma(x' + vt'), \qquad y = y'$$

$$z = z', \qquad t = \gamma\left(t' + \frac{v}{c^2}x'\right).$$

It should be noted that this inverse has the same form as the original except that the primed and unprimed variables are interchanged, and v is replaced by $-v$.

Since space and time are both subject to transformation, the basic unit is now an *event*, specified by a location in space and by its time of occurrence. Lorentz transformations always refer to events.

We now consider some elementary consequences of Lorentz transformations.

1. Length Contraction (Lorentz–Fitzgerald Contraction)

Suppose a rigid rod of length $L_0 = x_2' - x_1'$ is carried at rest in the frame K'. What is the length as measured in K? This length is equal to $L = x_2 - x_1$, where x_2 and x_1 are the positions of the ends of the rod *at the same time* t in the frame K. Thus we have the result

$$L_0 = x_2' - x_1' = \gamma(x_2 - x_1) = \gamma L,$$

$$L = \left(1 - \frac{v^2}{c^2}\right)^{1/2} L_0. \tag{4.3}$$

The rod appears shorter by a factor $\gamma^{-1} = (1 - v^2/c^2)^{1/2}$. The effect is really symmetric between the two observers. If the rod were carried by K, then K' would see its length contracted. How then can both take place together? If both carry rods (of the same length when compared at rest—say, meter sticks) each thinks the other's rod has shrunk! The point here is that each observer would object to the manner in which the other has carried out the measurement, since it would appear to each that the two ends of the moving stick were not marked *at the same time* by the other observer. This accounts for the apparent lack of symmetry implied by the contraction. (Since the Lorentz transformation of time depends on position, temporal simultaneity is not Lorentz invariant.)

2. Time Dilation

Suppose a device (clock) at rest at the origin of K' measures off an interval of time $T_0 = t_2' - t_1'$. What is the interval of time measured in K? Note that in K', the device moves so that $x' = 0$. Thus we obtain

$$T = t_2 - t_1 = \gamma(t_2' - t_1') = \gamma T_0. \tag{4.4}$$

The interval measured has increased by a factor $\gamma = (1 - v^2/c^2)^{-1/2}$, so that the moving clock appears to have slowed down. Again, the effect is symmetrical between the two observers: K' thinks clocks in K have slowed down, too. The resolution of this apparent contradiction is again a result of looking at the manner of measuring an interval of time between two events separated in space. K measures t_1 as the moving clock passes x_1, then measures t_2 as it passes x_2; he simply subtracts $t_2 - t_1$ on the assumption that his own two clocks at x_1 and x_2 are *synchronized.* K' will object to this, since according to his observations the two clocks in K are not synchronized at all.

In both the time-dilation and length-contraction effects we can see the powerful role played by the questions of synchronization of clocks and of the whole concept of simultaneity. Many of the apparent contradictions of special relativity are simply a result of the *relativity of simultaneity* between two events separated in space.

3. Transformation of Velocities

If a point has velocity $\mathbf{u}'$ in frame K', what is its velocity $\mathbf{u}$ in frame K (Fig. 4.2)? Writing Lorentz transformations for differentials [cf. Eqs. (4.2)]

$$dx = \gamma(dx' + v\,dt'), \qquad dy = dy'$$

$$dz = dz', \qquad dt = \gamma\left(dt' + \frac{v}{c^2}\,dx'\right).$$

We then have the relations

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + v\,dt')}{\gamma(dt' + v\,dx'/c^2)} = \frac{u'_x + v}{1 + vu'_x/c^2}, \tag{4.5a}$$

$$u_y = \frac{u'_y}{\gamma(1 + vu'_x/c^2)}, \tag{4.5b}$$

$$u_z = \frac{u'_z}{\gamma(1 + vu'_x/c^2)}. \tag{4.5c}$$

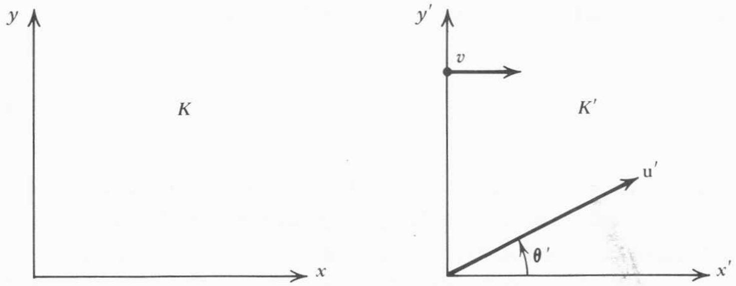


Figure 4.2 Lorentz transformation of velocities.

The generalization of these equations to an arbitrary velocity **v**, not necessarily along the x axis, can be stated in terms of the components of **u** perpendicular to and parallel to **v**:

$$u_{\parallel}=\frac{u'_{\parallel}+v}{\left(1+vu'_{\parallel}/c^2\right)}, \qquad u_{\perp}=\frac{u'_{\perp}}{\gamma\left(1+vu'_{\parallel}/c^2\right)}. \tag{4.6}$$

The directions of the velocities in the two frames are related by the *aberration formula,*

$$\tan\theta=\frac{u_{\perp}}{u_{\parallel}}=\frac{u'\sin\theta'}{\gamma(u'\cos\theta'+v)}, \tag{4.7}$$

where $u'\equiv|\mathbf{u}'|$. The azimuthal angle ϕ remains unchanged. An interesting application is for the case $u'=c$, where

$$\tan\theta=\frac{\sin\theta'}{\gamma(\cos\theta'+v/c)}, \tag{4.8a}$$

$$\cos\theta=\frac{\cos\theta'+v/c}{1+(v/c)\cos\theta'}. \tag{4.8b}$$

Equations (4.8) represent the *aberration of light.*

It is instructive to set $\theta'=\pi/2$, that is, a photon is emitted at right angles to v in K'. Then we have

$$\tan\theta=\frac{c}{\gamma v}, \tag{4.9a}$$

$$\sin\theta=\frac{1}{\gamma}. \tag{4.9b}$$

Now for highly relativistic speeds, $\gamma\gg 1$, θ becomes small:

$$\theta\sim\frac{1}{\gamma}. \tag{4.10}$$

If photons are emitted isotropically in K', then half will have $\theta'<\pi/2$ and half $\theta'>\pi/2$ (see Fig. 4.3). Equation (4.10) shows that in frame K photons are concentrated in the forward direction, with half of them lying within a cone of half-angle $1/\gamma$. Very few photons will be emitted having $\theta\gg 1/\gamma$. This is called the *beaming effect*.

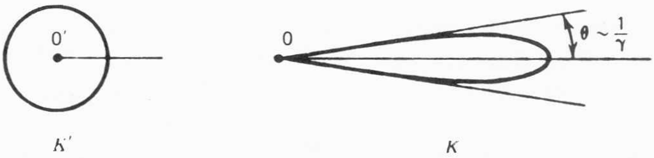


Figure 4.3 Relativistic beaming of radiation emitted isotropically in the rest frame K'.

4. Doppler Effect

We have seen that any periodic phenomenon in the moving frame K' will appear to have a longer period by a factor γ when viewed by local observers in frame K. If, on the other hand, we measure the *arrival times* of pulses or other indications of the periodic phenomenon that propagate with the velocity of light, then there will be an additional effect on the observed period due to the delay times for light propagation. The joint effect is called the *Doppler effect.*

In the rest frame of the observer K imagine that the moving source emits one period of radiation as it moves from point 1 to point 2 at velocity v. If the frequency of the radiation in the rest frame of the source is ω' then the time taken to move from point 1 to point 2 in the observer's frame is given by the time-dilation effect:

$$\Delta t = \frac{2\pi\gamma}{\omega'}$$

Now consider Fig. 4.4 and note $l = v\,\Delta t$ and $d = v\,\Delta t\cos\theta$. The difference in arrival times Δt_A of the radiation emitted at 1 and 2 is equal to Δt minus the time taken for radiation to propagate a distance d. Thus we have

$$\Delta t_A = \Delta t - \frac{d}{c} = \Delta t\left(1 - \frac{v}{c}\cos\theta\right).$$

Therefore, the observed frequency ω will be

$$\omega = \frac{2\pi}{\Delta t_A} = \frac{\omega'}{\gamma\left(1 - \frac{v}{c}\cos\theta\right)}. \tag{4.11}$$

This is the relativistic Doppler formula. The factor γ^{-1} is purely a relativistic effect, whereas the $1-(v/c)\cos\theta$ factor appears even classically. One distinction between the classical and relativistic points of view should be mentioned, however. The classical Doppler effect (say, for sound

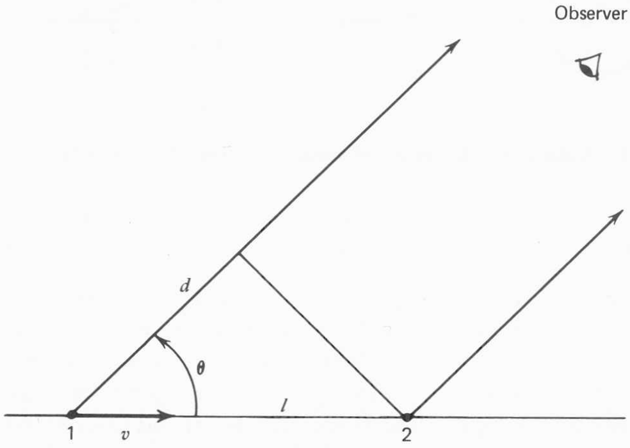


Figure 4.4 ***Geometry for the Doppler effect.***

waves) requires knowledge not only of the relative velocity between source and observer but also the velocities of source and observer relative to the medium (say, air) carrying the waves. The relativistic formula has no reference to an underlying medium for the propagation of light, and only the relative velocity of source and observer appears.

We can also write the Doppler formula as

$$\omega' = \omega\gamma\left(1 - \frac{v}{c}\cos\theta\right). \tag{4.12a}$$

It is easy to show that the inverse of this is

$$\omega = \omega'\gamma\left(1 + \frac{v}{c}\cos\theta'\right). \tag{4.12b}$$

5. Proper Time

Although intervals of space and time are separately subject to Lorentz transformation and thus have differing values in differing frames of reference, there are some quantities that are the same in all Lorentz frames. An important such *Lorentz invariant* is the quantity $d\tau$ defined by

$$c^2 d\tau^2 = c^2 dt^2 - \left(dx^2 + dy^2 + dz^2\right). \tag{4.13}$$

This is called the *proper time* element between the events differing by dx, dy, dz in space and dt in time. It is easily shown from Eqs. (4.2) that $d\tau$ is left unchanged under Lorentz transformations, $d\tau = d\tau'$.

The quantity $d\tau$ is called a proper time interval, because it measures time intervals between events occurring at the same spatial location ($dx = dy = dz = 0$), that is, ticks of clocks carried by an observer, which measure his own time.

If the coordinate differentials refer to the position of the origin of another reference frame traveling with velocity v, then

$$d\tau = dt\left(1 - \frac{v^2}{c^2}\right)^{1/2}. \tag{4.14}$$

Equation (4.14) is just the time dilation formula (4.4) in which $d\tau$ is the time interval measured by the observer in motion.

4.2 FOUR-VECTORS

We could continue to find Lorentz transformation properties of physical quantities using ad hoc methods, as in the preceding sections. However, a great deal of order can be brought to this task by introducing the concept of *four-vectors*. A four-vector has transformation properties that are identical to the transformation of coordinates of events [Eq. (4.2)]. Once it is established that a certain quantity is a four-vector, its transformation properties are fully defined. Most physical quantities can be related to four-vectors or to their generalizations—the tensors. It is easy to construct invariants from vectors and tensors, and in this way a physical result can often be obtained without using the Lorentz transformation at all.

The squared length of the three-dimensional vector $\mathbf{x}$, namely, $x^2 + y^2 + z^2$, is an invariant with respect to three-dimensional rotations. By analogy, the invariance of the quantity $s^2 = -c^2\tau^2 = -c^2t^2 + x^2 + y^2 + z^2$ suggests that the quantities x, y, z and t can be formed into a vector in a four-dimensional space, and that Lorentz transformations correspond to rotations in this space. Let us define

$$\begin{aligned} x^0 &= ct \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z. \end{aligned} \tag{4.15}$$

The quantities x^μ for $\mu=0,1,2,3$ define coordinates of an event in *space-time*. Just as x, y, and z form the components of a three-dimensional spatial vector **x**, we shall say that x^μ are the components of a four-dimensional space-time vector $\vec{x}$, or simply a *four-vector*.

The fact that the expression for s^2 contains a minus sign in front of c^2t^2 means that space-time is not a Euclidean space; it is a special space called *Minkowski space*. Such a space can be handled in two ways, either by including $\sqrt{-1}$ in the definition of the time component or by the introduction of a *metric*. Although the former method has some simplifying features, the latter method lends itself to the transition to general relativity, and so we adopt it here. Once the notational difficulties of the metric approach are mastered, it is not much more complicated than the $\sqrt{-1}$ approach.

Let us define the *Minkowski metric*. In Cartesian coordinates, the components of $\eta_{\mu\nu}$ are:

$$\eta_{\mu\nu}=\eta^{\mu\nu}=\begin{cases}-1, & \text{if } \mu=\nu=0\\ +1, & \text{if } \mu=\nu=1,2,3\\ 0 & \text{if } \mu\neq\nu.\end{cases} \tag{4.16a}$$

The distinction between superscripted and subscripted indices is explained shortly. The metric $\eta_{\mu\nu}$ can be presented as the 4×4 array (matrix):

$$\eta_{\mu\nu}=\eta^{\mu\nu}=\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{4.16b}$$

Note that this metric is symmetric: $\eta_{\mu\nu}=\eta_{\nu\mu}$. The invariant $s^2=-c^2t^2+x^2+y^2+z^2$ can now be written in terms of the metric:

$$s^2=\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\eta_{\mu\nu}x^\mu x^\nu. \tag{4.17a}$$

An important and beautiful notational advance (originated by Einstein) is the *summation convention*: In any single term containing a repeated Greek index, a summation is implied over that index with values 0, 1, 2, and 3. Therefore, we can write Eq. (4.17a) without the summation signs, since both μ and ν are repeated, once in $\eta_{\mu\nu}$ and then in x^μ or x^ν:

$$s^2=\eta_{\mu\nu}x^\mu x^\nu. \tag{4.17b}$$

We shall henceforth use the summation convention unless otherwise stated.

A few remarks should be made about the summation convention. Since a repeated index is summed over, its exact designation is irrelevant. Therefore, it is often called a *dummy index*, and any Greek letter can be used for it. Equation (4.17b) can also be written $s^2 = \eta_{\sigma\tau} x^\sigma x^\tau$, for example. Another point is that an index cannot be repeated more than twice in a single term; for example, the combination $\eta_{\mu\mu} x^\mu$ is regarded as meaningless.

An equivalent way to use the Minkowski metric is to define another set of components of the vector $\vec{x}$, denoted by x_μ, where

$$\begin{aligned} x_0 &= -ct, \\ x_1 &= x, \\ x_2 &= y, \\ x_3 &= z. \end{aligned} \tag{4.18}$$

These differ from the superscripted components x^μ only in the sign of the time component. The superscripted components are called the *contravariant components*, and the subscripted components are called the *covariant components*. The relation between the two can be written

$$x_\mu = \eta_{\mu\nu} x^\nu, \tag{4.19a}$$

$$x^\mu = \eta^{\mu\nu} x_\nu. \tag{4.19b}$$

Thus the metric can be used to *raise* or *lower* indices. Now the invariant s^2 can be written simply

$$s^2 = x^\mu x_\mu.$$

(Summation on indices occurs only between contravariant and covariant indices. As we show later, this ensures Lorentz invariance.)

The Lorentz transformation (4.2) (corresponding to a *boost* along the x axis) can be written simply in terms of a set of coefficients defined by the array (with $\beta \equiv v/c$)

$$\Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.20}$$

Then Eq. (4.2) can be written

$$x'^{\mu}=\Lambda^{\mu}{}_{\nu}x^{\nu}. \tag{4.21}$$

In fact, any arbitrary Lorentz transformation in Cartesian coordinates can be written in the form (4.21), since the spatial three-dimensional rotations necessary to align the x axes before and after the boost are also of linear form. The coefficients $\Lambda^{\mu}{}_{\nu}$ of such an arbitrary Lorentz transformation will, in general, not be given by Eq. (4.20), but will be more complicated.

The transformation law (4.21) defines the transformation of the contravariant components of the vector $\vec{x}$. Since the transformation must leave the quantity s^2 invariant, we must have

$$\eta_{\mu\nu}x^{\mu}x^{\nu}=\eta_{\sigma\tau}x'^{\sigma}x'^{\tau}=\eta_{\sigma\tau}\Lambda^{\sigma}{}_{\mu}\Lambda^{\tau}{}_{\nu}x^{\mu}x^{\nu}.$$

This can be true for arbitrary x^{μ} only if

$$\eta_{\mu\nu}=\Lambda^{\sigma}{}_{\mu}\Lambda^{\tau}{}_{\nu}\eta_{\sigma\tau}. \tag{4.22}$$

This equation can be regarded as the condition on the coefficients $\Lambda^{\mu}{}_{\nu}$ that yields the most general kind of Lorentz transformation. The transformations of interest to us are of a more restrictive nature, however. Note that Eq. (4.22) can be written in matrix form as $\eta=\Lambda^{T}\eta\Lambda$, where Λ^{T} is the transpose matrix of Λ. Taking determinants of this yields the result that $\det\Lambda=\pm 1$. We restrict ourselves to *proper* Lorentz transformations, for which

$$\det\Lambda=+1. \tag{4.23a}$$

This rules out reflections, such as $x\to -x$, that would change a right-handed coordinate system into a left-handed one. We also assume *isochronous* Lorentz transformations, for which

$$\Lambda^{0}{}_{0}\geqslant 1, \tag{4.23b}$$

so that the sense of flow of time is the same in K and K'. Note that the boost (4.20) satisfies both (4.23a) and (4.23b).

The transformation of the covariant components x_{μ} of the vector can be deduced from Eq. (4.19b) and (4.21):

$$x'_{\mu}=\tilde{\Lambda}_{\mu}{}^{\nu}x_{\nu} \tag{4.24}$$

where the coefficients $\tilde{\Lambda}_\mu{}^\nu$ are simply related to the $\Lambda^\mu{}_\nu$ by

$$\tilde{\Lambda}_\mu{}^\nu = \eta_{\mu\tau}\Lambda^\tau{}_\sigma\eta^{\sigma\nu}. \tag{4.25}$$

From the invariance of $s^2 = x^\mu x_\mu$ we easily deduce

$$\Lambda^\sigma{}_\nu\tilde{\Lambda}_\sigma{}^\mu = \delta^\mu{}_\nu, \tag{4.26}$$

where we have introduced the *Kronecker-δ*:

$$\delta^\mu{}_\nu = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu. \end{cases} \tag{4.27}$$

These are the components of the 4×4 unit matrix, which accounts for the substitution property of $\delta^\mu{}_\nu$: For any arbitrary quantity Q^μ we have

$$Q^\mu = \delta^\mu{}_\nu Q^\nu.$$

Note the useful result

$$\eta^{\mu\sigma}\eta_{\sigma\nu} = \delta^\mu{}_\nu. \tag{4.28}$$

Multiplying Eq. (4.21) by $\tilde{\Lambda}_\mu{}^\alpha$ and using Eq. (4.26) yields the inverse transformation:

$$x^\alpha = \tilde{\Lambda}_\mu{}^\alpha x'^\mu. \tag{4.29}$$

Everything so far has referred to the vector $\vec{x}$ alone. We now wish to define a general *four-vector* $\vec{A}$ as having four contravariant components A^μ in each Lorentz frame, such that the transformation of components between any two frames is given by the same transformation law as applies to x^μ, namely, Eq. (4.21):

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu. \tag{4.30}$$

The covariant components of $\vec{A}$ are found from the equation analogous to Eq. (4.19a),

$$A_\mu = \eta_{\mu\nu}A^\nu. \tag{4.31}$$

These transform according to

$$A'_\mu = \tilde{\Lambda}_\mu{}^\nu A_\nu. \tag{4.32}$$

Let us consider another four-vector $\vec{B}$ having covariant components B_μ, which transform like $B'_\mu = \tilde{\Lambda}_\mu{}^\sigma B_\sigma$. Multiplying this equation by Eq. (4.30) yields, with the use of Eq. (4.26),

$$A'^\mu B'_\mu = \Lambda^\mu{}_\nu \tilde{\Lambda}_\mu{}^\sigma A^\nu B_\sigma = \delta^\sigma{}_\nu A^\nu B_\sigma = A^\nu B_\nu .$$

Thus the *scalar product* of $\vec{A}$ and $\vec{B}$,

$$\vec{A}\cdot\vec{B} = A^\nu B_\nu = A'^\nu B'_\nu \tag{4.33}$$

is a *Lorentz invariant* or *scalar*. In particular, the "square" of a vector $\vec{A}^2 = A^\mu A_\mu$ is an invariant. Thus our starting point, the invariance of $\vec{x}^2$, is seen to be a general property of four-vectors. We should point out that in Minkowski space, where the metric is not wholly positive, it is possible for the "square" of a four-vector to be positive, zero, or even negative; these possibilities are associated with what are called, respectively, a *spacelike*, *null*, or *timelike* four-vector.

The zeroth component of any four-vector $\vec{A}$ is called the *time-component* A^0, while the first, second, and third form an ordinary three-vector $\mathbf{A}$, called the *space-components*. Often it is convenient to use latin indices to describe the space part, so that these always range over the values 1, 2, and 3. For example, we write

$$\vec{A}\cdot\vec{B} = -A^0 B^0 + \mathbf{A}\cdot\mathbf{B} = -A^0 B^0 + A^i B_i . \tag{4.34}$$

Three-vectors are always denoted by a boldfaced symbol, whereas four-vectors are denoted by an arrow over the symbol. It should be understood, however, that the division of a four-vector into spatial and time components is dependent on the coordinate system. It is clear that a boost will mix these parts, although spatial rotations will not; for this reason the division will only depend on the velocity of the frame of reference but not on its orientation.

Let us introduce some physically interesting four-vectors other than the prototype $\vec{x}$. First of all we see that the difference between the coordinates of two different events $x_2^\mu - x_1^\mu$ is also a vector, since each term transforms by the same linear transformation. In particular, the difference between two infinitesimally neighboring events dx^μ constitutes a four-vector. Dividing now by the invariant $d\tau$ clearly also yields a four-vector, the *four-velocity* $\vec{U}$, for which

$$U^\mu \equiv \frac{dx^\mu}{d\tau} . \tag{4.35}$$

The zeroth component of this is

$$U^0=\frac{dx^0}{d\tau}=\frac{c\,dt}{d\tau}=c\gamma_u, \tag{4.36a}$$

where $\gamma_u=(1-u^2/c^2)^{-1/2}$, and u is the magnitude of the ordinary velocity $\mathbf{u}=d\mathbf{x}/dt$. The spatial components are

$$U^i=\frac{dx^i}{d\tau}=\gamma_u u^i. \tag{4.36b}$$

We may write

$$\vec{U}=\gamma_u\binom{c}{\mathbf{u}}. \tag{4.37}$$

Thus the spatial part of $\vec{U}$ is γ_u times the *ordinary* velocity, whereas the time component is γ_u times c. In this way we have promoted the ordinary velocity into a four-vector. The transformation of U^μ under the boost (4.20) is

$$\begin{aligned}U'^0&=\gamma(U^0-\beta U^1),\\ U'^1&=\gamma(-\beta U^0+U^1),\\ U'^2&=U^2,\\ U'^3&=U^3.\end{aligned}$$

With the above definitions we have

$$\begin{aligned}\gamma_{u'}c&=\gamma(c\gamma_u-\beta\gamma_u u^1),\\ \gamma_{u'}u'^1&=\gamma(-\beta c\gamma_u+\gamma_u u^1),\\ \gamma_{u'}u'^2&=\gamma_u u^2,\\ \gamma_{u'}u'^3&=\gamma_u u^3.\end{aligned}$$

The first two of these are

$$\gamma_{u'}=\gamma\gamma_u(1-vu^1/c^2), \tag{4.38a}$$

$$\gamma_{u'}u'^1=\gamma\gamma_u(u^1-v). \tag{4.38b}$$

Since $u^1=u\cos\theta$, we obtain the transformation for speed in terms of the γ's:

$$\gamma_{u'}=\gamma\gamma_u\left(1-\frac{uv}{c^2}\cos\theta\right). \tag{4.39}$$

Dividing (4.38b) by (4.38a) yields the previously derived formula (4.5a):

$$u'^1 = \frac{u^1 - v}{1 - vu'/c^2}.$$

The "length" of $\vec{U}$ is found from

$$\vec{U}\cdot\vec{U} = U^\mu U_\mu = -(\gamma_u c)^2 + (\gamma_u \mathbf{u})^2 = -c^2, \tag{4.40}$$

which is clearly Lorentz invariant.

The four-velocity takes a particularly simple form in a frame in which the ordinary velocity $\mathbf{u}$ vanishes (the rest frame). In that case, we have

$$U'^\mu = c\begin{bmatrix}1\\0\\0\\0\end{bmatrix}. \tag{4.41}$$

Only the time component is nonzero. This property makes $\vec{U}$ a useful tool in picking out the time component of an arbitrary vector as measured by an observer with four-velocity $\vec{U}$:

$$A'^0 = -\frac{1}{c}U'_\mu A'^\mu.$$

But since $U'_\mu A'^\mu = \vec{U}\cdot\vec{A}$ is an invariant, we can write generally

$$A'^0 = -\frac{1}{c}\vec{U}\cdot\vec{A}, \tag{4.42}$$

where $\vec{U}\cdot\vec{A}$ can be evaluated in *any* convenient frame, not necessarily the rest frame. Two examples of this formula can be checked immediately: First, set $\vec{A} = \vec{U}$, and we obtain the trivial result $U'^0 = c$. Set $\vec{A} = \vec{x}$, and we find

$$x'^0 = -\frac{1}{c}x_\mu \frac{dx^\mu}{d\tau} = -\frac{1}{2c}\frac{d}{d\tau}(x_\mu x^\mu)$$

$$= -\frac{1}{2c}\frac{d}{d\tau}(-c^2\tau^2) = c\tau,$$

which is correct, since the proper time is physically equal to the time of a clock in the rest frame.

Another four-vector can be introduced by the following indirect arguments: An electromagnetic wave of plane type has space and time dependence proportional to $\exp(i\mathbf{k}\cdot\mathbf{x}-i\omega t)$. The phase of this wave must be an invariant to all observers, since the vanishing of the electric and magnetic fields in one frame implies their vanishing in all frames. (A charged particle moving on an unaccelerated straight-line trajectory in one frame must have such a trajectory in all frames, by the relativity principle.) Notice that we may write

$$\mathbf{k}\cdot\mathbf{x}-\omega t = k_\mu x^\mu = \vec{k}\cdot\vec{x},$$

where

$$k^\mu = \begin{pmatrix} \omega/c \\ \mathbf{k} \end{pmatrix}. \tag{4.43}$$

It can be shown easily that since the product $\vec{k}\cdot\vec{x}$ is an invariant and $\vec{x}$ is an arbitrary four-vector, then $\vec{k}$ must be a four-vector also. Therefore, we can write the transformation for $\vec{k}$ immediately

$$k'^0 = \gamma(k^0 - \beta k^1), \tag{4.44a}$$

$$k'^1 = \gamma(-\beta k^0 + k^1), \tag{4.44b}$$

$$k'^2 = k^2, \tag{4.44c}$$

$$k'^3 = k^3. \tag{4.44d}$$

Since $|\mathbf{k}| = \omega/c$ for electromagnetic waves, we have $k^1 = (\omega/c)\cos\theta$, so that the zeroth component of the transformation reduces to the Doppler formula

$$\omega' = \omega\gamma\left(1 - \frac{v}{c}\cos\theta\right). \tag{4.45}$$

Another way of deriving (4.45) is to apply (4.42) with $A^\mu = k^\mu$.

Note that $\vec{k}$ is a null vector, since

$$\vec{k}\cdot\vec{k} = |\mathbf{k}|^2 - \frac{\omega^2}{c^2} = 0 \tag{4.46}$$

where the last quantity vanishes by Eq. (2.20a).

The construction of four-vectors is by no means an automatic procedure, as our experience so far has shown. In two cases (x^μ and k^μ) we have simply used a known three-vector for the spatial part and added an appropriate time component. In one case (U^{μ}) we had to multiply by an appropriate factor γ_u to make the resultant a four-vector. In some cases to be treated presently (electric and magnetic fields) there is *no* four-vector that corresponds to a given three-vector. The systematic construction of four-vectors is best accomplished by means of *tensor analysis*, which we now consider.

4.3 TENSOR ANALYSIS

We are already familiar with some kinds of tensors: A *zeroth-rank tensor* is precisely what we have been calling a Lorentz invariant or Lorentz scalar. A *first-rank tensor* is precisely what we have been calling a four-vector.

Let us now define a *second-rank tensor*. The contravariant components of such a tensor, say T, are given by the sixteen numbers $T^{\mu\nu}$, where, as usual, μ and ν take on the values 0, 1, 2, and 3. The defining transformation properties of T are given by

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\tau}T^{\sigma\tau}. \tag{4.47}$$

We can define an associated set of covariant components $T_{\mu\nu}$ by lowering indices with the Minkowski metric

$$T_{\mu\nu} = \eta_{\mu\sigma}\eta_{\nu\tau}T^{\sigma\tau}. \tag{4.48}$$

It is easy to show that these components transform as

$$T'_{\mu\nu} = \tilde{\Lambda}_{\mu}{}^{\sigma}\tilde{\Lambda}_{\nu}{}^{\tau}T_{\sigma\tau}. \tag{4.49}$$

It is also possible to define *mixed components* such as

$$T^{\mu}{}_{\nu} = \eta_{\nu\tau}T^{\mu\tau}, \qquad T_{\mu}{}^{\nu} = \eta_{\mu\sigma}T^{\sigma\nu}. \tag{4.50}$$

These have the transformation properties

$$T'^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\sigma}\tilde{\Lambda}_{\nu}{}^{\tau}T^{\sigma}{}_{\tau}, \tag{4.51a}$$

$$T'_{\mu}{}^{\nu} = \tilde{\Lambda}_{\mu}{}^{\sigma}\Lambda^{\nu}{}_{\tau}T_{\sigma}{}^{\tau}. \tag{4.51b}$$

The position of the tensor index, as a superscript or subscript, determines whether it is contravariant or covariant in its transformations.

Since second-rank tensors are perhaps less familiar than vectors, let us give several examples:

1. The sixteen quantities $A^\mu B^\nu$, formed from the components A^μ and B^ν of two vectors. This can be proved by multiplying the transformation laws for the vector components:

$$A'^\mu B'^\nu = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau A^\sigma B^\tau$$

This is precisely of the form (4.47).

2. The Minkowski metric $\eta^{\mu\nu}$. The transformation of components of the second rank $\eta^{\mu\nu}$ transform by

$$\eta'^{\alpha\beta} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta^{\mu\nu}.$$

By comparison with Eq. (4.26), $\eta'^{\alpha\beta} = \eta^{\alpha\beta}$. Thus $\eta^{\alpha\beta}$ has the same components in all frames, as we have assumed.

3. The Kronecker–delta δ_ν^μ. A proof similar to the preceding one for the metric can be given starting with Eq. (4.26). This shows that δ_ν^μ forms the components of a mixed second-rank tensor.

Higher-rank tensors can be defined in a similar fashion. The transformation law involves a factor Λ for each contravariant index and a factor $\tilde{\Lambda}$ for each covariant index.

There are a number of simple and useful rules of *tensor analysis* that can be used to form tensors from other tensors:

1. *Addition*. Two tensors of the same type, having the same free indices, can be added to form another tensor of that same type. Examples: $A^\mu + B^\mu$; $F^\mu{}_\nu + G^\mu{}_\nu$. The proof follows from the linearity of the transformations.

2. *Multiplication*. Given two tensors having distinct free indices, multiplication will yield a tensor of rank equal to the sum of the ranks of the two tensors. Examples: $A^\mu B^\nu$ is a second-rank tensor; also $F^{\mu\nu}G_{\sigma\tau}$ is a fourth-rank tensor. The general proof follows the lines outlined above for $A^\mu B^\nu$.

3. *Raising and Lowering Indices*. The Minkowski metric can be used to change contravariant indices into covariant ones, and vice versa, by the processes of *raising* and *lowering*. For example, see Eqs. (4.19),

(4.31), and (4.48). The proof of this result depends on the results

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\sigma}=\tilde{\Lambda}_{\nu}{}^{\tau}\eta_{\tau\sigma}, \tag{4.52a}$$

$$\eta^{\mu\nu}\tilde{\Lambda}_{\mu}{}^{\sigma}=\Lambda^{\nu}{}_{\tau}\eta^{\tau\sigma}, \tag{4.52b}$$

which follow from Eqs. (4.25) and (4.28). This means the lowering operator $\eta_{\mu\nu}$, in commuting with the Lorentz transformation coefficients Λ, changes them to $\tilde{\Lambda}$, and this changes a contravariant index into a covariant one. A similar statement holds for the raising operator $\eta^{\mu\nu}$.

4. *Contraction*. Consider a tensor having at least two indices, one of which is contravariant and the other covariant. If these two indices are set equal, implying a summation over that index, then the result is a tensor of rank two less. Examples: The scalar product of two vectors $A^{\mu}B_{\mu}$ can be regarded as the contraction of the second-rank tensor $A^{\mu}B_{\nu}$. If $T^{\mu\nu}{}_{\sigma}$ is a third-order tensor, then $T^{\mu\nu}{}_{\nu}$ is a vector. Note that contraction can be used more than once in a single term. Thus starting with the fourth-rank tensor $F^{\mu\nu}G_{\sigma\tau}$ we can form the invariant $F^{\mu\nu}G_{\mu\nu}$. Let us prove this property of contraction for the above example of $T^{\mu\nu}{}_{\nu}$. From the transformation law for $T^{\mu\nu}{}_{\sigma}$ we obtain

$$T'^{\mu\nu}{}_{\nu}=\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\tilde{\Lambda}_{\nu}{}^{\tau}T^{\alpha\beta}{}_{\tau}.$$

But $\Lambda^{\nu}{}_{\beta}\tilde{\Lambda}_{\nu}{}^{\tau}=\delta^{\tau}_{\beta}$ [cf Eq. (4.26)], so that

$$T'^{\mu\nu}{}_{\nu}=\Lambda^{\mu}{}_{\alpha}T^{\alpha\beta}{}_{\beta},$$

showing that $T^{\mu\nu}{}_{\nu}$ is indeed a vector. The general proof of this property follows along similar lines.

5. *Gradients of Tensor Fields*. A *tensor field* is defined as a tensor that is a function of the spacetime coordinates x^0,x^1,x^2,x^3. Then the gradient operation $\partial/\partial x^{\mu}$ acting on such a field produces a tensor field of one higher rank with μ as a new *covariant* index. A convenient notation for the gradient operation is a comma followed by the index μ. Thus, for example, if λ is a scalar, then $\lambda_{,\mu}\equiv\partial\lambda/\partial x^{\mu}$ is a covariant vector. Similarly $\mathrm{T}^{\mu\nu}{}_{,\sigma}\equiv\partial\tau^{\mu\nu}/\partial x^{\sigma}$ is a third-rank tensor. We shall prove this rule for the special case of the vector field A^{μ}. Differentiating the transformation

$$A'^{\mu}=\Lambda^{\mu}{}_{\sigma}A^{\sigma},$$

gives

$$\frac{\partial A'^{\mu}}{\partial x'^{\nu}}=\Lambda^{\mu}{}_{\sigma}\frac{\partial A^{\sigma}}{\partial x'^{\nu}}=\Lambda^{\mu}{}_{\sigma}\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{\partial A^{\sigma}}{\partial x^{\alpha}}=\Lambda^{\mu}{}_{\sigma}\tilde{\Lambda}_{\nu}{}^{\alpha}\frac{\partial A^{\sigma}}{\partial x^{\alpha}},$$

where we have used Eq. (4.29) to evaluate $\partial x^{\alpha}/\partial x'^{\nu}$. This is recognized as the transformation for a second-rank tensor with contravariant index μ and covariant index ν. Note that we have assumed that the components of Λ are constant, a result in Cartesian coordinate systems but not in general (e.g., spherical) coordinate systems. In non-Cartesian systems, partial derivatives do not form the components of a tensor.

The above rules of tensor analysis are extremely useful in practice. Once they have been mastered they become almost automatic; the notation itself almost provides sufficient guidance as to the correct forms. In this regard we note that although the summation convention allows summation over any two indices, only when it involves a subscript-superscript pair is the result assured as a tensor. (See Problem 4.5.) Thus we have always been careful to define quantities with superscripts and subscripts in such a way as to satisfy this requirement.

Some further definitions concerning tensors follows: Tensors of second rank $T^{\mu\nu}$ are *symmetric* or *antisymmetric* if $T^{\mu\nu}=T^{\nu\mu}$ or if $T^{\mu\nu}=-T^{\nu\mu}$, respectively. The *divergence* of a tensor field is a gradient followed by a contraction of the gradient index with one of the other contravariant indices; For example, $A^{\mu}{}_{,\mu}\equiv$ divergence of the vector A^{μ}; $T^{\mu\nu}{}_{,\nu}\equiv$ divergence of the tensor $T^{\mu\nu}$.

A *tensor equation* is a statement that two tensors of the same rank and type are equal. A fundamental property of a tensor equation is that *if it is true in one Lorentz frame, then it is true in all Lorentz frames*. This is clearly true, since each side transforms in the same way. For this reason tensor equations automatically obey the postulate of relativity, which makes them an ideal way to state the laws of nature. This property of the equations of physics under Lorentz transformation is called *invariance of form* or *Lorentz covariance* or simply *covariance*. (This use of the word "covariance" has nothing to do with covariant components of tensors.) Covariance plays a powerful role in helping decide what the proper equations of physics are; in the next section we see this role clearly.

4.4 COVARIANCE OF ELECTROMAGNETIC PHENOMENA

It is empirically found that Maxwell's equations are valid in all Lorentz frames. The two parameters that enter Maxwell's equations and the Lorentz force equation are c and e, the velocity of light and charge, respectively. If Maxwell's equations are to be Lorentz invariant in form, then c and e must be Lorentz scalars; c is invariant by one of the postulates of special relativity. Also, it is an empirical fact that e is invariant. If ρ is a charge density, then $de=\rho\, dx_1\, dx_2\, dx_3$ is a Lorentz invariant. But the four-volume element $dx_0\, dx_1\, dx_2\, dx_3$ is an invariant, since the Jacobian of the transformation from x_μ to x'_μ is simply the determinant of Λ, which has been shown [Eq. (4.23a)] to be unity. Thus ρ must transform in the same manner as the zeroth component of a four-vector.

To find the other three components, note that the equation of charge conservation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{j}=0$$

can be written as a tensor equation,

$$j^{\mu}{}_{,\mu}=0, \tag{4.53}$$

where $\vec{j}$ has components

$$j^{\mu}=\begin{pmatrix}\rho c\\ \mathbf{j}\end{pmatrix}. \tag{4.54}$$

This four-vector is called the *four-current*.

We next look at the set of vector and scalar wave equations in the Lorentz gauge, Eqs. (2.66):

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\frac{4\pi}{c}\mathbf{j},$$

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=-4\pi\rho.$$

If we define the *four-potential*

$$A^{\mu}=\begin{pmatrix}\phi\\ \mathbf{A}\end{pmatrix}, \tag{4.55}$$

then the wave equations may be written as the tensor equations

$$A^{\beta,\alpha}_{,\alpha}=-\frac{4\pi}{c}j^{\beta}. \tag{4.56}$$

The Lorentz gauge

$$\nabla\cdot\mathbf{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}=0,$$

should be preserved under Lorentz transformations, since it was used to obtain the tensor equations (4.56). Indeed, it can be written as a scalar equation,

$$A^{\alpha}{}_{,\alpha}=0. \tag{4.57}$$

What is the tensor representing the fields themselves, **E** and **B**? Since these fields are obtained from derivatives of **A** and ϕ, they should be expressible in terms of derivatives of the four-potential $A_{\mu,\nu}$. Since **E** and **B** have six components all together, we consider the *antisymmetric* tensor

$$F_{\mu\nu}\equiv A_{\nu,\mu}-A_{\mu,\nu}, \tag{4.58}$$

because a rank two antisymmetric tensor has exactly six independent components. From the relationship between the fields and potentials, (2.58) and (2.60), we may write the components as

$$F_{\mu\nu}=\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{4.59}$$

To check that $F_{\mu\nu}$ is the object we want, let us see that it can be used to write Maxwell's equations in tensor form: The two Maxwell equations containing sources,

$$\nabla\cdot\mathbf{E}=4\pi\rho, \qquad \nabla\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=\frac{4\pi}{c}\mathbf{j}$$

can be written as

$$F_{\mu\nu}{}^{,\nu}=\frac{4\pi}{c}j_{\mu}, \tag{4.60}$$

as can easily be checked. Note that Eq. (4.53), (4.56), (4.57), and (4.60) all involve tensor divergences. The conservation of charge, Eq. (4.53), easily follows from Eq. (4.60):

$$\frac{4\pi}{c}j^{\mu}{}_{,\mu}=F^{\mu\nu}{}_{,\mu\nu}=0,$$

where the last relation follows from the fact that

$$F^{\mu\nu}{}_{,\mu\nu}=-F^{\nu\mu}{}_{,\mu\nu}=-F^{\nu\mu}{}_{,\nu\mu}.$$

The "internal" Maxwell equations,

$$\nabla\cdot\mathbf{B}=0, \qquad \nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0,$$

can be written as

$$F_{\mu\nu,\sigma}+F_{\sigma\mu,\nu}+F_{\nu\sigma,\mu}=0. \tag{4.61a}$$

This equation can be written concisely as

$$F_{[\mu\nu,\sigma]}=0, \tag{4.61b}$$

where [] around indices denote all permutations of indices, with even permutations contributing with a positive sign and odd permutations with a negative sign, for example,

$$A_{[\alpha\beta]}=A_{\alpha\beta}-A_{\beta\alpha}. \tag{4.62}$$

Using the same notation, we can write

$$F_{\mu\nu}=A_{[\nu,\mu]}. \tag{4.63}$$

Since $F_{\mu\nu}$ is a second-rank tensor, its components transform in the usual way, that is,

$$F'_{\mu\nu}=\tilde{\Lambda}_{\mu}{}^{\alpha}\tilde{\Lambda}_{\nu}{}^{\beta}F_{\alpha\beta}. \tag{4.64}$$

Using this transformation law and the definition of $F_{\mu\nu}$ we obtain the transformation law for the fields **E** and **B**. For a pure boost with velocity $\mathbf{v}=c\boldsymbol{\beta}$, these equations can be written in the form:

$$\mathbf{E}'_{\parallel}=\mathbf{E}_{\parallel} \qquad \mathbf{B}'_{\parallel}=\mathbf{B}_{\parallel} \tag{4.65a}$$

$$\mathbf{E}'_{\perp}=\gamma(\mathbf{E}_{\perp}+\boldsymbol{\beta}\times\mathbf{B}) \quad \mathbf{B}'_{\perp}=\gamma(\mathbf{B}_{\perp}-\boldsymbol{\beta}\times\mathbf{E}). \tag{4.65b}$$

One immediate consequence of these equations is that the concept of a pure electric or pure magnetic field is not Lorentz invariant. If the field is purely electric ($\mathbf{B}=0$) in one frame, in another frame it will be, in general, a mixed electric and magnetic field. Thus the general term *electromagnetic* field.

Any scalar formed from $F_{\mu\nu}$ represents a function of **E** and **B** which is a Lorentz invariant. One such scalar is just the dot product of F with itself, or "square" of F

$$F_{\mu\nu}F^{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2). \tag{4.66}$$

Thus $\mathbf{B}^2 - \mathbf{E}^2 = \mathbf{B}'^2 - \mathbf{E}'^2$ is invariant under Lorentz transformations. Another scalar which can be obtained from F is just the determinant of F:

$$\det F = (\mathbf{E}\cdot\mathbf{B})^2. \tag{4.67}$$

Thus $\mathbf{E}\cdot\mathbf{B} = \mathbf{E}'\cdot\mathbf{B}'$ is also an invariant. It is easy to show that the determinant of any second-rank tensor is a scalar, since

$$\begin{aligned}\det A_{\alpha\beta} &= \det A'_{\mu\nu}\tilde{\Lambda}_{\alpha}{}^{\mu}\tilde{\Lambda}_{\beta}{}^{\nu} = (\det\tilde{\Lambda})^2 \det A'_{\mu\nu} \\ &= \det A'_{\mu\nu}.\end{aligned}$$

4.5 A PHYSICAL UNDERSTANDING OF FIELD TRANSFORMATIONS

It is sometimes useful to understand Lorentz transformations of quantities in terms of a piecemeal intuitive approach, as well as in terms of the elegant language of tensor transformations. For example, by means of a simple physical model we can derive the transformation of the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ represented in Eqs. (4.65) for the case of an initially pure electric field ($\mathbf{B}=0$). Consider a charged capacitor with plates perpendicular to the x axis in its rest frame K'. Let σ be the surface charge density (esu/cm^2). Then it is known that the electric field inside is $E = 4\pi\sigma$, independent of the separation of the plates d and has a direction normal to the plates.

In frame K' the capacitor is moving with velocity v and the plates are separated by d/γ. The surface charge density is unchanged $\sigma' = \sigma$, because the net charge on a surface element is invariant, and the surface area of the element is also invariant, because the y and z components are unchanged. Since the field depends only on surface charge density and not on plate separation we have $E' = E$, so that in general we have

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}$$

as we had previously found.

Now consider the capacitor turned so that the plates are perpendicular to the y axis. The charge density σ is now increased by a factor γ because of length contraction, and we also have a surface current density of magnitude $\mu' = -\sigma' v$, which gives rise to a magnetic field in the z direction of magnitude $B_z' = -(4\pi/c)\mu'$. Thus for this case we have

$$\mathbf{E}'_\perp = \gamma \mathbf{E}_\perp, \qquad \mathbf{B}'_\perp = -\gamma \boldsymbol{\beta} \times \mathbf{E}_\perp.$$

It is also possible to treat the case of an initially pure magnetic field by a similar model, and thus to derive Eqs. (4.65) by superposition. However, we omit the details here.

4.6 FIELDS OF A UNIFORMLY MOVING CHARGE

Let us apply Eqs. (4.65) to find the fields of a charge moving with constant velocity v along the x axis. In the rest frame of the particle the fields are

$$E_x' = \frac{qx'}{r'^3} \qquad B_x' = 0$$

$$E_y' = \frac{qy'}{r'^3} \qquad B_y' = 0$$

$$E_z' = \frac{qz'}{r'^3} \qquad B_z' = 0$$

where

$$r'^3 = (x'^2 + y'^2 + z'^2)^{3/2}.$$

The inverse of the transformation of the fields Eq. (4.65) is simply found by intercharging primed and unprimed quantities and reversing the sign of v. Then it follows that

$$E_x = \frac{qx'}{r'^3} \qquad B_x = 0$$

$$E_y = \frac{q\gamma y'}{r'^3} \qquad B_y = -\frac{q\gamma\beta z'}{r'^3}$$

$$E_z = \frac{q\gamma z'}{r'^3} \qquad B_z = \frac{q\gamma\beta y'}{r'^3}.$$

These are given in terms of the primed coordinates. We can Lorentz transform the coordinates to give

$$E_x = \frac{q\gamma(x-vt)}{r^3} \qquad B_x = 0$$
$$E_y = \frac{q\gamma y}{r^3} \qquad B_y = -\frac{q\gamma\beta z}{r^3} \tag{4.68}$$
$$E_z = \frac{q\gamma z}{r^3} \qquad B_z = \frac{q\gamma\beta y}{r^3}$$

where

$$r^3 = \left[\gamma^2(x-vt)^2 + y^2 + z^2\right]^{3/2}.$$

Now, we may show that Eqs. (4.68) are precisely what one obtains from the fields given by the Liénard–Wiechert potentials Eqs. (3.7a) and (3.7b). To do this, let us first find where the retarded position of the particle is. For simplicity, assume $z=0$. Then we have (Fig. 4.5)

$$t_{\text{ret}} = t - \frac{R}{c}$$
$$R^2 = y^2 + (x - vt_{\text{ret}})^2$$
$$= y^2 + \left(x - vt + \frac{vR}{c}\right)^2.$$

Solving for R, we obtain

$$R = \gamma^2\beta\bar{x} + \gamma\left(y^2 + \gamma^2 x^2\right)^{1/2},$$

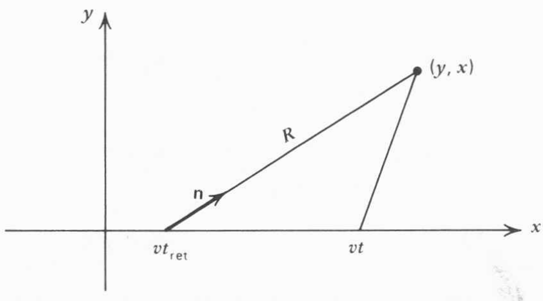


Figure 4.5 ***Evaluation of the radiation field from the retarded position of the particle.***

where

$$\bar{x} \equiv x - vt.$$

We can write the unit vector **n** as

$$\mathbf{n} = \frac{y\hat{\mathbf{y}} + (x - vt + vR/c)\hat{\mathbf{x}}}{R} \tag{4.69a}$$

and κ as:

$$\begin{aligned}\kappa &= 1 - \mathbf{n}\cdot\boldsymbol{\beta} \\ &= \frac{(y^2 + \gamma^2\bar{x}^2)^{1/2}}{\gamma R}.\end{aligned}$$

Thus we have the result

$$\frac{q}{\gamma^2 R^2 \kappa^3} = \frac{\gamma R q}{(y^2 + \gamma^2\bar{x}^2)^{3/2}}. \tag{4.69b}$$

Using Eqs. (4.69a) and (4.69b), and Eq. (4.68), we find that

$$\mathbf{E} = q\left[\frac{(\mathbf{n} - \boldsymbol{\beta})(1 - \beta^2)}{\kappa^3 R^2}\right],$$

which is identical to the field components of Eq. (3.10).

An important application of these results is the case of a highly relativistic charge, $\gamma \gg 1$. For simplicity, let us choose the field point to be a distance b from the origin along the y axis; this involves no loss in generality (see Fig. 4.6). Then we have the results

$$E_x = -\frac{qv\gamma t}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} \qquad B_x = 0 \tag{4.70a}$$

$$E_y = \frac{q\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} \qquad B_y = 0 \tag{4.70b}$$

$$E_z = 0 \qquad B_z = \beta E_y. \tag{4.70c}$$

For large γ we have $\beta \approx 1$ and $E_y \approx B_z$. In Fig. 4.7 E_x and E_y are plotted as functions of time. We see that the fields are strong only when t is of the same order as $b/\gamma v$. This means that the fields of the moving charge are

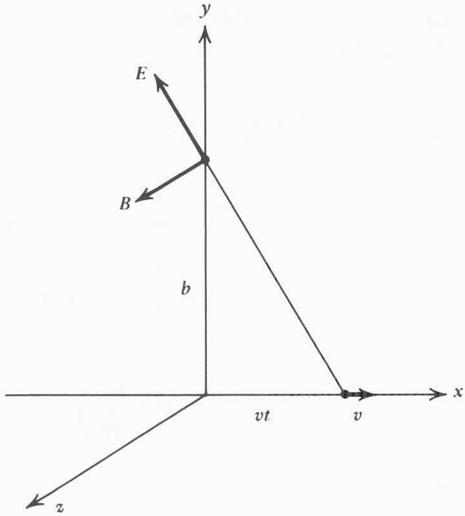


Figure 4.6 Electric and magnetic fields from a uniformly moving particle.

concentrated in the plane transverse to its motion, in fact, into an angle of order $1/\gamma$. The fields are also mostly transverse, since E_x is at maximum only of order q/b^2. Therefore, the field of a highly relativistic charge appears to be a pulse of radiation traveling in the same direction as the charge and confined to the transverse plane. This connection between the fields of a highly relativistic charge and an associated radiation field is an important one and is used in the *method of virtual quanta*, to be discussed in Chapter 5.

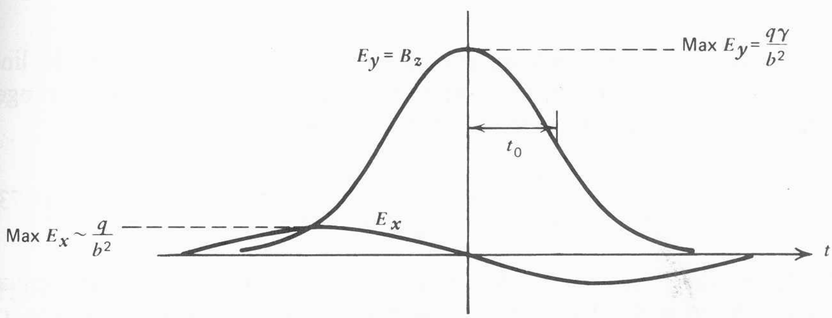


Figure 4.7 Time-dependence of fields from a particle of uniform high velocity.

We can determine the equivalent spectrum of this pulse of virtual radiation. First we must find the transform

$$\begin{aligned}\hat{E}(\omega) &= \frac{1}{2\pi}\int E_2(t)e^{i\omega t}\,dt \\ &= \frac{q\gamma b}{2\pi}\int_{-\infty}^{\infty}(\gamma^2 v^2 t^2 + b^2)^{-3/2}e^{i\omega t}\,dt. \qquad (4.71)\end{aligned}$$

This integral can be done in terms of the modified Bessel function of order one, K_1:

$$\hat{E}(\omega) = \frac{q}{\pi b v}\frac{b\omega}{\gamma v}K_1\left(\frac{b\omega}{\gamma v}\right). \qquad (4.72a)$$

Thus the spectrum is

$$\frac{dW}{dA\,d\omega} = c|\hat{E}(\omega)|^2 = \frac{q^2 c}{\pi^2 b^2 v^2}\left(\frac{b\omega}{\gamma v}\right)^2 K_1^2\left(\frac{b\omega}{\gamma v}\right). \qquad (4.72b)$$

The spectrum starts to cut off for $\omega > \gamma v/b$, which we could have predicted on the basis of the uncertainty principle, since the pulse is confined roughly to a time interval of order $b/\gamma v$. In fact, the complete behavior of $\hat{E}(\omega)$ can be estimated to within a factor of ~ 2 just by analysis of the picture of $E(t)$: $E(t)$ has a maximum $q\gamma/b^2$ for a time interval $\sim b/\gamma v$. Thus we approximate

$$\hat{E}_{\max}(\omega) \sim E_{\max}(t)\,\Delta t \sim \left(\frac{q\gamma}{b^2}\right)\left(\frac{b}{\gamma v}\right),$$

$$\Delta\omega \sim \frac{1}{\Delta t} \sim \frac{\gamma v}{b}.$$

We have found the spectrum per unit area at a distance b from the line of the charge's motion. To find the total energy per unit frequency range, we must integrate this over $dA = 2\pi b\,db$ (see Fig. 4.8):

$$\frac{dW}{d\omega} = 2\pi\int_{b_{\min}}^{b_{\max}}\frac{dW}{dA\,d\omega}\,b\,db. \qquad (4.73)$$

The lower limit has been chosen not as zero but as some minimum distance $b_{\min}$, such that the approximation of the field by means of classical electrodynamics and a point charge is valid. Two possible choices

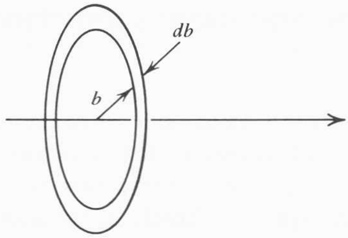


Figure 4.8 Area element perpendicular to the velocity of a moving particle.

are (1) $b_{\min}$ = radius of ion, if field is that of an ion and (2) $b_{\min} \sim \hbar/mc$ = Compton wavelength of particle. The integral is now

$$\frac{dW}{d\omega} = \frac{2q^2c}{\pi v^2}\int_x^\infty yK_1^2(y)\,dy, \tag{4.74a}$$

where

$$y \equiv \frac{\omega b}{\gamma v}, \qquad x \equiv \frac{\omega b_{\min}}{\gamma v}.$$

This integral can be done in terms of Bessel functions

$$\frac{dW}{d\omega} = \frac{2q^2c}{\pi v^2}\left[xK_0(x)K_1(x) - \tfrac{1}{2}x^2\left(K_1^2(x) - K_0^2(x)\right)\right]. \tag{4.74b}$$

Two limiting forms occur when ω is small, $\omega \ll \gamma v/b_{\min}$, and when ω is large, $\omega \gg \gamma v/b_{\min}$:

$$\frac{dW}{d\omega} = \frac{2q^2c}{\pi v^2}\ln\left(\frac{0.68\gamma v}{\omega b_{\min}}\right), \qquad \omega \ll \frac{\gamma v}{b_{\min}} \tag{4.75a}$$

$$\frac{dW}{d\omega} = \frac{q^2c}{2v^2}\exp\left(-\frac{2\omega b_{\min}}{\gamma v}\right), \qquad \omega \gg \frac{\gamma v}{b_{\min}}. \tag{4.75b}$$

These forms can be derived approximately by direct integration of $xK_1^2(x)$, using the asymptotic results $K_1(x) \sim 1/x$, $x \ll 1$, and $K_1(x) \sim (\pi/2x)^{1/2}e^{-x}$, $x \gg 1$.

4.7 RELATIVISTIC MECHANICS AND THE LORENTZ FOUR-FORCE

The equations of electrodynamics came to us in the already covariant form of Maxwell's equations. Unfortunately, the equations of dynamics as given by Newton are not in convariant form; this is clear since they obey Galilean not Lorentz invariance. Therefore, we must find new equations that reduce to the Newtonian ones for low velocities but that obey the principles of relativity. To do this we are guided by the requirement that these equations be cast in covariant, tensor form.

The rest mass of a particle m_0 is a scalar by definition, since it can be invariantly defined (go to a frame in which the particle is at rest and measure it). Then the *four-momentum* of a particle, P^μ is defined by

$$P^\mu \equiv m_0 U^\mu. \tag{4.76}$$

In the nonrelativistic limit, the spatial components of the four-momentum are just the components of the ordinary three-momentum, $m_0\mathbf{v}$. To interpret all the components relativistically, consider the expansion of P^0c for $v \ll c$:

$$P^0 c = m_0 c U^0 = m_0 c^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = m_0 c^2 + \tfrac{1}{2} m_0 v^2 + \cdots. \tag{4.77}$$

The second term in (4.77) is the nonrelativistic expression for the kinetic energy of the particle; therefore, we interpret $E = P^0 c$ as the total energy of the particle. The quantity $m_0 c^2$, being independent of v, is interpreted as the rest energy of the particle. If the relativistic expression for the spatial momentum is then defined as $\mathbf{p} = \gamma_v m_0 \mathbf{v}$, then $P^\mu = (E/c, \mathbf{p})$. Then from Eqs. (4.40), (4.76) and (4.77):

$$\vec{P}^2 = -m_0^2 c^2 = -\frac{E^2}{c^2} + |\mathbf{p}|^2,$$

$$E^2 = m_0^2 c^4 + c^2 |\mathbf{p}|^2. \tag{4.78}$$

Since photons are massless and travel at the speed of light, the four-momentum cannot be defined by Eq. (4.76). In this case we still define $P^\mu = (E/c, \mathbf{p})$, but we use the quantum relations $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$. From Eq. (4.43) we then have

$$P^\mu = \hbar k^\mu = \begin{pmatrix} \hbar\omega/c \\ \hbar\mathbf{k} \end{pmatrix}. \tag{4.79}$$

The momentum four-vector for photons is null, $\vec{P}^2 = 0$, since $E = |\mathbf{p}|c$.

Now, in exactly the same way as we obtained the four-velocity from the displacement four-vector, we can define a *four-acceleration* a^μ by taking another derivative, with respect to the scalar interval, of the four-velocity:

$$a^\mu \equiv \frac{dU^\mu}{d\tau}. \tag{4.80}$$

In the nonrelativistic limit, in which $\gamma_u \approx 1$, the spatial components of the four-velocity and four-acceleration are approximately equal to their nonrelativistic, three-vector counterparts.

Note that the four-acceleration and four-velocity are *orthogonal* (their dot product vanishes):

$$\begin{aligned}\vec{a}\cdot\vec{U} &= \frac{dU^\mu}{d\tau}U_\mu = \tfrac{1}{2}\frac{d}{d\tau}(U^\mu U_\mu)\\ &= \tfrac{1}{2}\frac{d}{d\tau}(-c^2) = 0.\end{aligned} \tag{4.81}$$

Having defined the four-acceleration, we can define another four-vector, the *four-force* F^μ, so as to obtain a relativistic form of Newton's equation "$F = ma$":

$$F^\mu \equiv m_0 a^\mu = \frac{dP^\mu}{d\tau}. \tag{4.82}$$

In the case of electromagnetism, we can explicitly evaluate F^μ from the known Lorentz force,

$$\mathbf{F}_{\text{Lorentz force}} = e\left[\mathbf{E} + \frac{1}{c}(\mathbf{v}\times\mathbf{B})\right].$$

Our Lorentz four-force should involve the electromagnetic fields embodied in the tensor $F_{\mu\nu}$ and the particle velocity embodied in the four-velocity U^μ and should also be a four-vector and proportional to the (scalar) charge of the body. The simplest possibility is

$$F^\mu = \frac{e}{c}F^\mu{}_\nu U^\nu. \tag{4.83}$$

Substituting Eq. (4.83) into Eq. (4.82), we have the tensor equation of motion of a charged particle:

$$a^\mu = \frac{e}{m_0 c}F^\mu{}_\nu U^\nu. \tag{4.84}$$

Let us check the components of Eq. (4.84) to see if it is indeed what we want. The $\mu=0$ component is, using Eqs. (4.59) and (4.76):

$$\frac{dW}{dt}=e\mathbf{E}\cdot\mathbf{v}. \tag{4.85a}$$

Equation (4.85a) is just the conservation of energy: the rate of change of particle energy W is the mechanical work done on the particle by the field, $e\mathbf{E}\cdot\mathbf{v}$. Each spatial component (say, $\mu=1$) of Eq. (4.84) is

$$\frac{dP_x}{dt}=e\left[E_x+\frac{1}{c}(\mathbf{v}\times\mathbf{B})_x\right], \tag{4.85b}$$

agreeing with the desired expression for the three-Lorentz force.

Note from Eq. (4.81) and Eq. (4.82) that the four-force, *regardless of its origin*, is always orthogonal to the four-velocity:

$$\vec{F}\cdot\vec{U}=m_0(\vec{a}\cdot\vec{U})=0. \tag{4.86}$$

Equation (4.86) is a general property of any covariant formulation of mechanics in four-dimensional spacetime. It implies that every four-force must have some velocity dependence, although this dependence might become negligible in the nonrelativistic limit. For the Lorentz four-force, in particular, we find

$$\vec{F}_{\text{Lorentz force}}\cdot\vec{U}=\frac{e}{c}F_{\mu\nu}U^{\mu}U^{\nu}=0,$$

because $F_{\mu\nu}$ is antisymmetric and $U^{\mu}U^{\nu}$ is symmetric.

4.8 EMISSION FROM RELATIVISTIC PARTICLES

Total Emission

We would now like to use relativistic transformations to find the radiation emitted by a particle moving at relativistic speeds. The idea is to move into an *instantaneous rest frame* K', such that the particle has zero velocity at a certain time. The particle will not remain at rest in this frame (since it can accelerate), but at least for infinitesimally neighboring times the particle moves nonrelativistically. We can therefore calculate the radiation emitted by use of the dipole (Larmor) formula. Suppose a total amount of energy dW' is emitted in this frame in time dt'. The momentum of this radiation is zero, $d\mathbf{p}'=0$, because the emission is symmetrical with respect to any direction and its opposite direction. The energy in a frame K moving with velocity $-v$ with respect to the particle is therefore

$$dW=\gamma\, dW',$$

from the transformation properties of the four-momentum. The time interval dt is simply

$$dt=\gamma\, dt',$$

since dt' is the proper time of the particle. The total power emitted in frames K and K' are given by

$$P=\frac{dW}{dt}, \qquad P'=\frac{dW'}{dt'}.$$

From above we see

$$P=P'. \tag{4.87}$$

Thus the total emitted power is a Lorentz invariant for any emitter that emits with front-back symmetry in its instantaneous rest frame. Knowing this, we would like to express the power in covariant form. Now, from the Larmor formula, we have [cf. Eq. (3.19)]

$$P'=\frac{2q^2}{3c^3}|\mathbf{a}'|^2. \tag{4.88}$$

Recall, however, that because $\vec{a}\cdot\vec{U}=0$ [cf. Eq. (4.81)], and because $U^\mu=(\mathrm{c},\mathbf{0})$ in the instantaneous rest frame of the emitting particle, [cf. Eq. (4.41)], we have

$$a_0'=0.$$

Thus

$$|\mathbf{a}'|^2=a_k'a'^k=a_0'a'^0+a_k'a'^k=a'^\alpha a_\alpha'=\vec{a}\cdot\vec{a}. \tag{4.89}$$

So, we can write Eq. (4.88) in manifestly covariant form:

$$P=\frac{2q^2}{3c^3}\vec{a}\cdot\vec{a}. \tag{4.90}$$

The power can thus be evaluated in any frame just by computing $\vec{a}$ in that particular frame and squaring it.

It is convenient to express P in terms of the three-vector acceleration $d^2\mathbf{x}/dt^2$ rather than in terms of the four-vector acceleration $d^2x^\mu/d\tau^2$. It can easily be shown (see Problem 4.3) that if K' is an instantaneous rest frame of a particle, then

$$a'_{\parallel}=\gamma^3 a_{\parallel}, \tag{4.91a}$$

$$a'_{\perp}=\gamma^2 a_{\perp}. \tag{4.91b}$$

Thus we can write

$$P=\frac{2q^2}{3c^3}\mathbf{a}'\cdot\mathbf{a}'=\frac{2q^2}{3c^3}\left(a'^2_{\parallel}+a'^2_{\perp}\right)$$

$$=\frac{2q^2}{3c^3}\gamma^4\left(a^2_{\perp}+\gamma^2 a^2_{\parallel}\right). \tag{4.92}$$

Angular Distribution of Emitted and Received Power

In the instantaneous rest frame of the particle, let us consider an amount of energy dW' that is emitted into the solid angle $d\Omega'=\sin\theta'\,d\theta'\,d\phi'$ about the direction at angle θ' to the x' axis (see Fig. 4.9). It is convenient to introduce the notations

$$\mu=\cos\theta, \qquad \mu'=\cos\theta',$$

so that

$$d\Omega=d\mu\,d\phi, \qquad d\Omega'=d\mu'\,d\phi'.$$

Since energy and momentum form a four-vector, the transformation of the energy of the radiation is,

$$dW=\gamma(dW'+v\,dP'_x)=\gamma(1+\beta\mu')\,dW'. \tag{4.93}$$

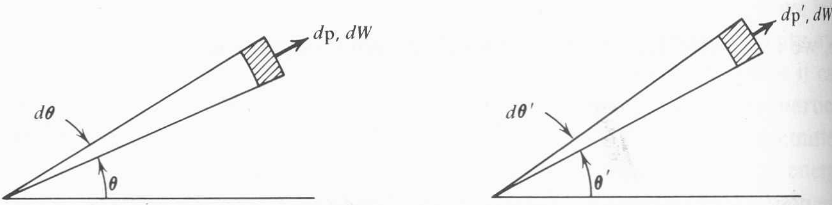


***Figure 4.9** Lorentz transformation of the angular distribution of emitted power.*

We also have from Eq. (4.8b),

$$\mu = \frac{\mu' + \beta}{1 + \beta\mu'}. \tag{4.94}$$

Differentiating this yields

$$d\mu = \frac{d\mu'}{\gamma^2(1+\beta\mu')^2},$$

and since $d\phi = d\phi'$,

$$d\Omega = \frac{d\Omega'}{\gamma^2(1+\beta\mu')^2}. \tag{4.95}$$

Thus we have the result

$$\frac{dW}{d\Omega} = \gamma^3(1+\beta\mu')^3 \frac{dW'}{d\Omega'}. \tag{4.96}$$

The power emitted in the rest frame P' is found simply by dividing dW' by the time interval dt'. However, in frame K there are *two* possible choices for the time interval used to divide dW:

1—$dt = \gamma\, dt'$. This is the time interval during which the emission occurs in frame K [cf. Eq. (4.4)]. With this choice we obtain the *emitted* power in frame $K: P_e$.

2—$dt_A = \gamma(1-\beta\mu)\, dt'$. This is the time interval of the radiation as received by a stationary observer in K. The extra factor is the retardation effect due to the moving source [cf. Eq. (4.11) and (4.12b)]. With this choice we obtain the *received* power in frame $K: P_r$.

Thus we obtain the two results:

$$\frac{dP_e}{d\Omega} = \gamma^2(1+\beta\mu')^3 \frac{dP'}{d\Omega'} = \frac{1}{\gamma^4(1-\beta\mu)^3}\frac{dP'}{d\Omega'}, \tag{4.97a}$$

$$\frac{dP_r}{d\Omega} = \gamma^4(1+\beta\mu')^4 \frac{dP'}{d\Omega'} = \frac{1}{\gamma^4(1-\beta\mu)^4}\frac{dP'}{d\Omega'}. \tag{4.97b}$$

The alternate forms follow from the equivalence of Eqs. (4.12a) and (4.12b).

Which of these two should we use? P_r is the power actually measured by an observer and so would seem to be the natural one. Also in favor of P_r is that Eq. (4.97b) has the expected symmetry property of yielding the inverse transformation by interchanging primed and unprimed variables, along with a change of sign of β. For these reasons we deal with P_r for the rest of this section, calling it simply P.

It should be pointed out, however, that P_e does have its uses (c.f. Jackson's Sect. 14.3; also our discussion of emission coefficient, §4.9). In practice, the distinction between emitted and received power is often not important, since P_r and P_e are equal in an average sense for stationary distributions of particles. We discuss this further in the context of synchrotron emission in §6.7.

Let us now return to Eq. (4.97b). If the radiation is isotropic in the particle's frame (or nearly isotropic), then the angular distribution in the observer's frame will be highly peaked in the forward direction for highly relativistic velocities ($\beta \sim 1$). In fact, let us write

$$\mu = \cos\theta \approx 1 - \frac{\theta^2}{2}, \tag{4.98a}$$

$$\beta = \left(1 - \frac{1}{\gamma^2}\right)^{1/2} \approx 1 - \frac{1}{2\gamma^2}. \tag{4.98b}$$

It follows by expansion that

$$\frac{1}{\gamma^4(1-\beta\mu)^4} \approx \left(\frac{2\gamma}{1+\gamma^2\theta^2}\right)^4. \tag{4.98c}$$

This latter factor is sharply peaked near $\theta \simeq 0$ with an angular scale of order $1/\gamma$, in agreement with our previous discussion.

Let us now apply these formulas to the case of an emitting particle. In the instantaneous rest frame of the particle the angular distribution is given by [cf. Eq. (3.18)]

$$\frac{dP'}{d\Omega'} = \frac{q^2 a'^2}{4\pi c^3} \sin^2\Theta',$$

where Θ' is the angle between the acceleration and the direction of emission (see Fig. 4.10). Writing $\mathbf{a}' = \mathbf{a}'_{\parallel} + \mathbf{a}'_{\perp}$ and using the results (4.91a) and (4.91b), we obtain

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} \frac{\left(\gamma^2 a_{\parallel}^2 + a_{\perp}^2\right)}{(1-\beta\mu)^4} \sin^2\Theta'. \tag{4.99}$$

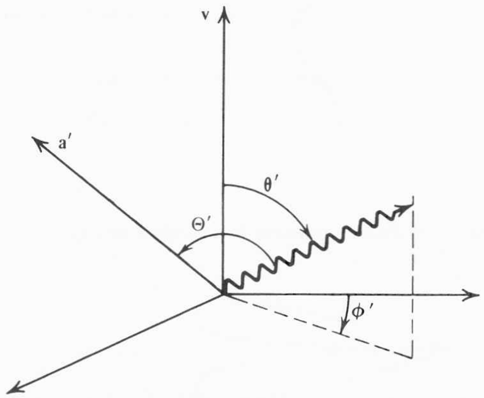


Figure 4.10 Geometry for dipole emission from a particle instantaneously at rest.

To use this formula we must relate Θ' to the angles in the frame K. This is difficult in the general case, so we work out the angular distribution of the received power for special cases:

1—Acceleration || to Velocity. Here $\Theta'=\theta'$ so that

$$\sin^2\Theta' = \frac{\sin^2\theta}{\gamma^2(1-\beta\mu)^2} \tag{4.100}$$

where we have used Eq. (4.94). Substituting Eq. (4.100) into Eq. (4.99) with $a_\perp=0$, we obtain

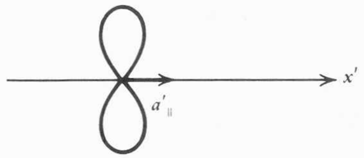
$$\frac{dP_\parallel}{d\Omega} = \frac{q^2}{4\pi c^3} a_\parallel^2 \frac{\sin^2\theta}{(1-\beta\mu)^6}. \tag{4.101}$$

2—Acceleration $\perp$ to Velocity. Here $\cos\Theta'=\sin\theta'\cos\phi'$, so that

$$\sin^2\Theta' = 1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\mu)^2}. \tag{4.102}$$

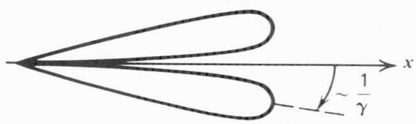
Thus we have the result

$$\frac{dP_\perp}{d\Omega} = \frac{q^2 a_\perp^2}{4\pi c^3} \frac{1}{(1-\beta\mu)^4}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\mu)^2}\right]. \tag{4.103}$$



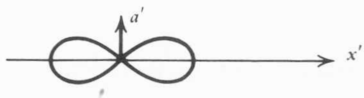
(a)

Figure 4.11a Dipole radiation pattern for particle at rest.



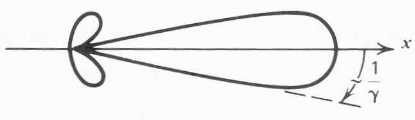
(b)

Figure 4.11b Angular distribution of radiation emitted by a particle with parallel acceleration and velocity.



(c)

Figure 4.11c Same as a.



(d)

Figure 4.11d Angular distribution of radiation emitted by a particle with perpendicular acceleration and velocity.

3—Extreme Relativistic Limit. When $\gamma \gg 1$, the quantity $(1-\beta\mu)$ in the denominators becomes small in the forward direction, and the radiation becomes strongly peaked in this direction. Using the same arguments as before, we obtain

$$(1-\beta\mu) \approx \frac{1+\gamma^2\theta^2}{2\gamma^2}.$$

For the parallel-acceleration case the received radiation pattern is

$$\frac{dP_{\|}}{d\Omega} \approx \frac{16q^2a_{\|}^2}{\pi c^3}\gamma^{10}\frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)^6}, \tag{4.104}$$

while for perpendicular acceleration,

$$\frac{dP_{\perp}}{d\Omega} \approx \frac{4q^2a_{\perp}^2}{\pi c^3}\gamma^{8}\frac{1-2\gamma^2\theta^2\cos 2\phi+\gamma^4\theta^4}{(1+\gamma^2\theta^2)^6}. \tag{4.105}$$

Both of these expressions depend on θ solely through the combination $\gamma\theta$. Therefore, the peaking is for angles $\theta \sim 1/\gamma$, which can be seen in Fig. 4.11, where polar diagrams of the radiation patterns are given.

4.9 INVARIANT PHASE VOLUMES AND SPECIFIC INTENSITY

Consider a group of particles that occupy a slight spread in position and in momentum at a particular time. In a frame comoving with the particles they occupy a spatial volume element $d^3\mathbf{x}' = dx'\,dy'\,dz'$ and a momentum volume element $d^3\mathbf{p}' = dP_x'\,dP_y'\,dP_z'$, but no spread in energy, $dW' = -dP_0' = 0$. This is because the contribution to the energy from the space momentum in the rest frame is quadratic and thus vanishes to the first order. The group thus occupies an element of phase space $d\mathcal{V}' = d^3\mathbf{p}'\,d^3\mathbf{x}'$. We now wish to show that any observer not comoving with the particles will conclude that they occupy the same amount of phase space in his frame $d\mathcal{V} = d^3\mathbf{p}\,d^3\mathbf{x}$. Thus a phase space element is Lorentz invariant.

Let the observer have velocity parameter β with respect to the comoving K' frame and orient axes so that he moves along the x axis. Consider first the spatial volume element $d^3\mathbf{x}$ occupied by the particles, as measured by K. Since perpendicular distances are unaffected, $dy = dy'$ and $dz = dz'$. But there is a length contraction in the x direction [cf. Eq. (4.3)], $dx = \gamma^{-1}dx'$, thus yielding the relation

$$d^3\mathbf{x} = \gamma^{-1}d^3\mathbf{x}'. \tag{4.106a}$$

Now consider the momentum volume element measured by the observer, $d^3\mathbf{p}$. The components of momentum transform as components of a four-vector, yielding $dP_y'\,dP_z' = dP_y\,dP_z$, $dP_x = \gamma(dP_x' + \beta dP_0')$. But since the particles have the same energy in the comoving frame, $dP_x = \gamma\,dP_x'$, and we obtain

$$d^3\mathbf{p} = \gamma\, d^3\mathbf{p}'. \tag{4.106b}$$

Combining Eqs. (4.106a) and (4.106b), we see that

$$d\mathcal{V} = d\mathcal{V}'. \tag{4.107a}$$

Since frames K and K' have arbitrary relative velocity, we have the result

$$d\mathcal{V} = \text{Lorentz invariant}. \tag{4.107b}$$

Equation (4.107a) was strictly derived only for particles of finite mass, so that frame K' could be a rest frame. However, no reference to particle mass occurs in Eq. (4.107b), and therefore it has applicability to the limiting case of photons.

From Eq. (4.107b), it follows simply that the phase space density

$$f = \frac{dN}{d\mathcal{V}} \tag{4.108}$$

is an invariant, since the number of particles within the phase volume element, dN, is a countable quantity and therefore itself invariant.

It is easy to relate the phase space density of photons to the specific intensity I_ν and thus determine the transformation properties of I_ν. This is done by evaluating the energy density per unit solid angle per frequency range in two ways, using f and also the quantity $u_\nu(\Omega)$, defined in §1.3:

$$h\nu f p^2\, dp\, d\Omega = U_\nu(\Omega)\, d\Omega\, d\nu. \tag{4.109}$$

Since $U_\nu(\Omega) = I_\nu / c$ and $p = h\nu / c$ we find that I_ν / ν^3 is simply proportional to the Lorentz invariant f, so that

$$\frac{I_\nu}{\nu^3} = \text{Lorentz invariant}. \tag{4.110}$$

Having determined the Lorentz transformation properties of the specific intensity, we should now like to determine the transformation properties of other transfer quantities. Because the source function occurs in the transfer equation as the difference $I_\nu - S_\nu$, it is clear that S_ν must have the same transformation properties as I_ν, namely,

$$\frac{S_\nu}{\nu^3} = \text{Lorentz invariant}. \tag{4.111}$$

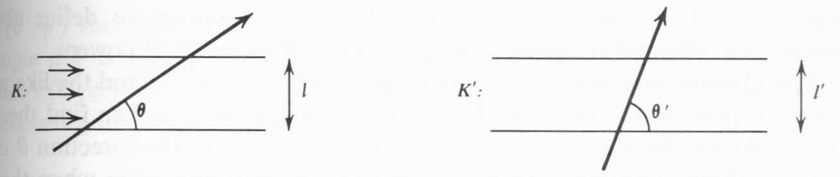


Figure 4.12 Transformation of a moving, absorbing medium.

To find the transformation of absorption coefficient we imagine material in frame K streaming with velocity v between two planes parallel to the x axis. Let K' be the rest frame of the material. (See Fig. 4.12). The optical depth τ along the ray must be an invariant, since $e^{-\tau}$ gives the *fraction of photons* passing through the material, and this involves simple counting. Thus we have the result

$$\tau = \frac{l\alpha_\nu}{\sin\theta} = \frac{l}{\nu\sin\theta}\nu\alpha_\nu = \text{Lorentz invariant.}$$

The transformation of $\sin\theta$ can be found by noting that $\nu\sin\theta$ is simply proportional to the y component of the photon four-momentum k_y. But both k_y and l are the same in both frames, being perpendicular to the motion. Therefore

$$\nu\alpha_\nu = \text{Lorentz invariant.} \tag{4.112}$$

Finally we find the transformation of the emission coefficient $j_\nu = \alpha_\nu S_\nu$ from Eqs. (4.111) and (4.112):

$$\frac{j_\nu}{\nu^2} = \text{Lorentz invariant.} \tag{4.113}$$

Another derivation of Eq. (4.113) can be based on Eq. (4.97a). The emission coefficient can be written as

$$j_\nu = n\frac{dP_e}{d\Omega\, d\nu}, \tag{4.114}$$

where n is the density of emitters (particles/cm^3). Now, from Eq. (4.12b) we have $d\nu = d\nu'\gamma(1+\beta\mu')$, and also $n = \gamma n'$ by Lorentz contraction along the motion. Thus we have

$$j_\nu = \gamma^2(1+\beta\mu')^2 n' \frac{dP'}{d\Omega'\, d\nu'} = \left(\frac{\nu}{\nu'}\right)^2 j'_\nu.$$

and Eq. (4.113) follows. Notice that here it is essential to define the emission coefficient in terms of *emitted* rather than *received* power.

It is often convenient to determine the quantities α_ν, j_ν, S_ν and the like in the rest frame of the material. By the above results we can then find them in any frame. Because the transformation of ν involves the direction θ of the ray, these quantities will not, in general, be isotropic, even when they are isotropic in the rest frame. The observed nonisotropy of the cosmic microwave background can be used to find the velocity of the earth through the background (c.f. Problem 4.13).

PROBLEMS

4.1—In astrophysics it is frequently argued that a source of radiation which undergoes a fluctuation of duration Δt must have a physical diameter of order $D \lesssim c\,\Delta t$. This argument is based on the fact that even if all portions of the source undergo a disturbance at the same instant and for an infinitesimal period of time, the resulting signal at the observer will be smeared out over a time interval $\Delta t_{\min} \sim D/c$ because of the finite light travel time across the source. Suppose, however, that the source is an optically thick spherical shell of radius $R(t)$ that is expanding with relativistic velocity $\beta \sim 1, \gamma \gg 1$ and energized by a stationary point at its center. By consideration of relativistic beaming effects show that if the observer sees a fluctuation from the shell of duration Δt at time t, the source may actually be of radius

$$R < 2\gamma^2 c\,\Delta t,$$

rather than the much smaller limit given by the nonrelativistic considerations. In the rest frame of the shell surface, each surface element may be treated as an isotropic emitter.

This latter argument has been used to show that the active regions in quasars may be much larger than $c\,\Delta t \sim 1$ light month across, and thus avoids much energy being crammed into so small a volume.

4.2—Suppose that an observer at rest with respect to the fixed distant stars sees an isotropic distribution of stars. That is, in any solid angle $d\Omega$ he sees $dN = N(d\Omega/4\pi)$ stars, where N is the total number of stars he can see.

Suppose now that another observer (whose rest frame is K') is moving at a relativistic velocity β in the x direction. What is the distribution of stars seen by this observer? Specifically, what is the distribution function

$P(\theta',\phi')$ such that the number of stars seen by this observer in his solid angle $d\Omega'$ is $P(\theta',\phi')\,d\Omega'$? Check to see that $\int P(\theta',\phi')\,d\Omega' = N$, and check that $P(\theta',\phi') = N/4\pi$ for $\beta = 0$. In what direction will the stars "bunch up," according to the moving observer?

4.3

a. Show that the transformation of acceleration is

$$a_x = \frac{a'_x}{\gamma^3\sigma^3},$$

$$a_y = \frac{a'_y}{\gamma^2\sigma^2} - \frac{u'_y v}{c^2}\frac{a'_x}{\gamma^2\sigma^3},$$

$$a_z = \frac{a'_z}{\gamma^2\sigma^2} - \frac{u'_z v}{c^2}\frac{a'_x}{\gamma^2\sigma^3},$$

where

$$\sigma \equiv 1 + \frac{v u'_x}{c^2}.$$

b. If K' is the instantaneous rest frame of the particle, show that

$$a'_{\parallel} = \gamma^3 a_{\parallel},$$

$$a'_{\perp} = \gamma^2 a_{\perp},$$

where $a_{\parallel}$ and $a_{\perp}$ are the components parallel and perpendicular to the direction of v, respectively.

4.4—A rocket starts out from earth with a constant acceleration of $1g$ in its own frame. After 10 years of its own (proper) time it reverses the acceleration, and in 10 more years it is again at rest with respect to the earth. After a brief time for exploring, the spacemen retrace their journey back to earth, completing the entire trip in 40 years of their own time.

a. Let t be earth time and x be the position of the rocket as measured from earth. Let τ be the proper time of the rocket and let $\beta = c^{-1}dx/dt$. Show that the equation of motion of the rocket during the first phase of positive acceleration is

$$\gamma^3\frac{d^2x}{dt^2} = g.$$

b. Integrate this equation to show that

$$\beta = \frac{gt/c}{\sqrt{(gt/c)^2+1}}.$$

c. Integrating again, show that

$$x = \frac{c^2}{g}\left[\sqrt{(gt/c)^2+1} - 1\right].$$

d. Show that the proper time is related to earth time by

$$\frac{gt}{c} = \sinh\left(\frac{g\tau}{c}\right)$$

so that

$$x = \frac{c^2}{g}\left[\cosh\left(\frac{g\tau}{c}\right) - 1\right].$$

e. How far away do the spacemen get?

f. How long does their journey last from the point of view of an earth observer? Will friends be there to greet them when they return?

Hint: In answering parts (**e**) and (**f**) you need only the results for the first positive phase of acceleration plus simple arguments concerning the other phases.

g. Answer parts (**e**) and (**f**) if the spacemen can tolerate an acceleration of $2g$ rather than $1g$.

4.5—Show that $A^{\alpha}B^{\alpha}$ is not in general a scalar, where A^{α} and B^{α} are four-vector components.

4.6—Suppose in some inertial frame K a photon has four-momentum components

$$P_{\mu} = (-E, E, 0, 0).$$

(We use units where $c=1$). There is a special class of Lorentz transformations—called the "little group of P"—which leave the components of P unchanged, for example, a pure rotation through an angle α in the y-z

plane,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} -E \\ E \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -E \\ E \\ 0 \\ 0 \end{bmatrix},$$

is such a transformation. Find a sequence of pure boosts and pure rotations whose product is *not* a pure rotation in the y-z plane, but *is* in the little group of P.

4.7—An object emits a blob of material at speed v at an angle θ to the line-of-sight of a distant observer (see Fig. 4.13).

a. Show that the apparent transverse velocity inferred by the observer (i.e., the angular velocity on the sky times the distance to the object) is

$$v_{\text{app}} = \frac{v \sin\theta}{1-(v/c)\cos\theta}.$$

b. Show that v_{app} can exceed c; find the angle for which v_{app} is maximum, and show that this maximum is $v_{\text{max}} = \gamma v$.

4.8—Let two different uniformly moving observers have velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, in units where $c=1$. Show that their relative velocity, as measured

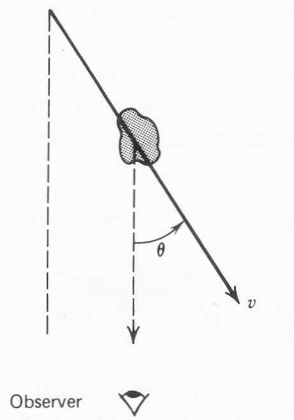


Figure 4.13 ***Emitting blob traveling at angle θ with respect to the line of sight.***

by one of the observers, satisfies

$$v^2 = \frac{(1-\mathbf{v}_1\cdot\mathbf{v}_2)^2-(1-v_1^2)(1-v_2^2)}{(1-\mathbf{v}_1\cdot\mathbf{v}_2)^2}.$$

A straight application of velocity transformations is painfully tedious, but an application of 4-vector invariants is trivial!

4.9—In ordinary three-space, Ohm's law is $\mathbf{j}=\sigma\mathbf{E}$ where $\mathbf{j}$ is the current, $\mathbf{E}$ the electric field, and σ the conductivity. Assuming σ is a scalar, write a four-tensor form of Ohm's law using the four-current j_μ, the Maxwell field tensor $F_{\mu v}$ and the four-velocity of the conducting element U_μ. Remember, a tensor equation that reduces to the correct expression in any frame (e.g., the rest frame of the conducting element) is correct in all frames.

4.10—A particle of rest mass m moves with velocity v in frame K. In its rest frame K' the particle emits some of its internal energy W' in the form of isotropic radiation.

a. Argue that there is no net reaction force on the particle and it remains at rest in K'.

b. What is the total momentum of the emitted radiation as seen in frame K?

c. Since this momentum is emitted into the forward direction, does the particle slow down as a result? If so, how can this be reconciled with the fact that the particle remains at rest in K'? If not, how can this be reconciled with the conservation of momentum?

4.11—A particle (rest mass m) initially at rest absorbs a photon of energy $h\nu$ and converts this energy into increased internal energy (say, heat). The particle has increased its rest mass to m' and moves with some velocity v'.

a. Setting up the conservation of energy and momentum, show that

$$\frac{m}{m'}=\left(1+\frac{2h\nu}{mc^2}\right)^{-1/2}.$$

b. By considering the appropriate Lorentz transformations, show that if the particle had been moving initially and absorbed a photon of energy $h\nu$, this same equation for the ratio of the initial and final rest masses holds with ν' replacing ν, where ν' is given by the Doppler formula.

4.12—Consider a particle of dust orbiting a star in a circular orbit, with velocity v. This particle absorbs stellar photons, heats up, and then emits the excess energy isotropically in its rest frame.

a. Show that in absorbing a photon the angular momentum of the particle about the star does not change. (Assume the photons are traveling radially outward from the star.)

b. When the particle emits its radiation, show that the velocity and its direction do not change, but that the angular momentum now decreases by the ratio m/m' of the rest mass after and before emission. Denoting the angular momenta before and after by l_0 and l, show that

$$l = l_0\left(1 + \frac{2h\gamma\nu}{mc^2}\right)^{-1/2}.$$

c. Having obtained this general result, let us now assume $v \ll c$ and $h\nu \ll mc^2$. By expanding, show that to lowest order the change in angular momentum caused by one photon is

$$\Delta l = -\frac{l_0 h\nu}{mc^2}.$$

Historical note: This result, although now for nonrelativistic particles, apparently cannot be derived classically. Attempts to do so by Poynting and others led to results differing from the correct answer by various numerical factors. Robertson resolved the problem in 1937 (*Mon. Not. Roy. Astron. Soc.* **97**, 423), showing that it is a relativistic effect even to lowest order. The above phenomenon is called the *Poynting–Robertson effect.*

d. A dust grain having a mass $m \sim 10^{-11}$g and cross section $\sigma \sim 10^{-8}$ cm^2 orbits the Sun at 1 A.U. Assuming that it always keeps a circular orbit, find the time for it to fall into the Sun.

4.13

a. Show that an observer moving with respect to a blackbody field of temperature T will see blackbody radiation with a temperature that depends on angle according to

$$T' = \frac{(1 - v^2/c^2)^{1/2}}{1 + (v/c)\cos\theta'}\,T.$$

b. The isotropy of the 2.7 K universal blackbody radiation at $\lambda = 3$ cm has been established to about one part in 10^3. What is the maximum velocity that the earth can have with respect to the frame in which this radiation is isotropic? [Isotropy is measured by the ratio $(I_{max} - I_{min})/(I_{max} + I_{min})$.] A positive result of this magnitude has recently been obtained.

4.14—A particle is accelerated by a force having components $F_{\parallel}$ and $F_{\perp}$ with respect to the particle's velocity. Show that the radiated power is

$$P = (2e^2/3m^2c^3)(F_{\parallel}^2 + \gamma^2 F_{\perp}^2).$$

Thus the perpendicular component has more effect in producing radiation that the parallel component by a factor γ^2.

4.15—Show that $U_{em}^2 - c^{-2}\mathbf{S}^2$ is a Lorentz scalar, where U_{em} is the free-space electromagnetic energy density and $\mathbf{S}$ is the Poynting vector.

4.16—Consider the *stress-energy tensor* for an electromagnetic field

$$T^{\mu\nu} \equiv \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)$$

where $F^{\alpha\beta}$ and $\eta^{\mu\nu}$ are the electromagnetic field tensor and Minkowski metric, respectively.

a. Show that $T^{\mu\nu}$ is traceless: $T^{\mu}{}_{\mu} = 0$.

b. Show that in free space $T^{\mu\nu}$ is divergenceless: $T^{\mu\nu}{}_{,\nu} = 0$.

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