MathMethods

Assignment 1.

1. a) use $\cos n\theta = \mathrm{Re}\,((\cos\theta + i\sin\theta)^n)$

use binomial expansion for $(\cos\theta + i\sin\theta)^n$ and pick out the real piece.

b) again use $\sum_{n=0}^{\infty} p^n \cos nx = \mathrm{Re}\left(\sum_{n=0}^{\infty} p^n e^{inx}\right) (= A)$

It is a geometric series.
So, $A = \mathrm{Re}\left(\frac{a}{1-r}\right)$ for $a=1$, $r = pe^{ix}$.

c) use the definition : $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

use the definition of exponential, only odd powers survive.
Do the similar thing for $\cos z$.

2) Both results follow trivially from CR cond^n.

a) $\partial_x^2 u = \partial_x \partial_y v$, $\partial_y^2 u = -\partial_y \partial_x v$

$\Rightarrow \nabla^2 u = 0$

Illy for $\nabla^2 v$.

b) $\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = \left(-\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right)$ using Cauchy Riemann eqn.

3) use CR eqn to get $\frac{\partial v}{\partial x} = 6xy$.

So that $v(x,y) = 3x^2y + f(y)$. — (1)

Here, ~~th~~ the integration "constant" for x-integral f can be a functn. of y.

another CR ~~set~~ of eqn $\Rightarrow \dfrac{\partial v}{\partial y} = 3x^2 - 3y^2$.

using ① in above eqn, fix $f(y)$.

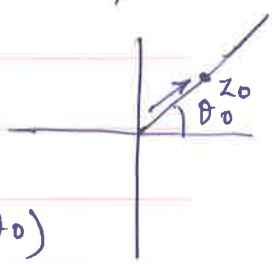
finally, $v = 3x^2y - y^3 + C$.

where C is independent of x, y.

4) w_1 is analytic $\Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

w_1^* is analytic $\Rightarrow \dfrac{\partial u}{\partial x} = -\dfrac{\partial v}{\partial y}$, $\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial y}$.

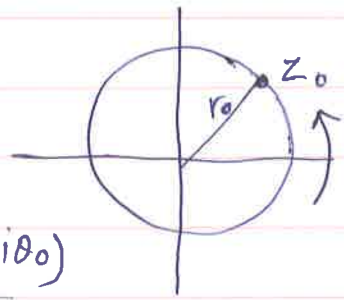
Hence, u, v are constants.



5) 1) Find derivative along a path $\theta = \theta_0$

$$\Rightarrow f'(z_0) = \lim_{r \to r_0} \frac{f(re^{i\theta_0}) - f(r_0 e^{i\theta_0})}{e^{i\theta_0}(r - r_0)}$$

$$= e^{-i\theta_0} \left. \frac{\partial f}{\partial r} \right|_{r_0}$$



2) along the path $|z| = r_0$

$$\Rightarrow f'(z_0) = \lim_{\theta \to \theta_0} \frac{f(r_0 e^{i\theta}) - f(r_0 e^{i\theta_0})}{r_0(e^{i\theta} - e^{i\theta_0})}$$

$$= \frac{e^{-i\theta_0}}{} \frac{\partial f}{} \Big|$$

use $\lim_{\theta \to \theta_0} \frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}} = \lim_{\theta \to \theta_0} \frac{1}{i e^{i\theta_0}}$.

Demand two expressions for $f'(z_0)$ are equal.

Assignment 2.

1) Take log of both sides.
you need to prove:

$$A = \cot\left(-\frac{i}{2} \ln \frac{ia-1}{ia+1}\right) = a.$$

use $\cot\theta = i \frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}}$.

So,

$$A = i \frac{\left(\frac{ia-1}{ia+1}\right)^{1/2} + \left(\frac{ia-1}{ia+1}\right)^{-1/2}}{\left(\frac{ia-1}{ia+1}\right)^{1/2} - \left(\frac{ia-1}{ia+1}\right)^{-1/2}}$$

Simplify this expression to ~~obt~~ prove the statement.

2) So, if $A(x,t) = A_0 \cancel{(x,t)}\, e^{i\omega(t - nx/c)}$

with $n \to n - ik$

$$A(x,t) = A_0 \cancel{(x,t)}\, e^{-kx/c}\, e^{i\omega(t-nx/c)}$$

The amplitude decays exponentially with distance.

3) To define $\partial_z f(z)$, we need, $f = u + iv$,

to satisfy, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

To define $\partial_{z^*} f(z)$,

with same logic we need, $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$.

$\Rightarrow$ ~~f~~ u, v are constant.

4) For contour $|z| = R$,

$$\frac{1}{2\pi i} \oint_C z^{m-n-1}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} (Re^{i\theta})^{m-n-1} d(Re^{i\theta})$$

$$= \frac{R^{m-n}}{2\pi i} \int_0^{2\pi} e^{i\theta(m-n-1)}\, i e^{i\theta}\, d\theta \quad -①$$

$$= R^{m-n} \left. \frac{e^{i\theta(m-n)}}{i(m-n)} \right|_0^{2\pi}$$

$\neq 0$ only if $m = n$.

if $m = n$, ① $= \frac{R^{m-m}}{2\pi} \int_0^{2\pi} d\theta = 1$

Hence, proved.

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5). $\oint_C \frac{dz}{z^2-1}$; C circle with $|z| = 2$.

$= \oint_C \frac{dz}{(z+1)(z-1)}$ $z = \pm 1$ are simple pole, both inside the contour.

(*) you can solve it by deforming the contour OR by Residue theorem.

The result : $2\pi i \left(\frac{1}{-2} + \frac{1}{2} \right) = 0$.

6) $\because$ $f(z)$ is continuous non zero every where inside C,

$w(z) = \frac{1}{f(z)}$ is also analytic inside C.

Now, we know, $w(z_0) = \frac{1}{2\pi i} \oint_C \frac{w(z)}{z - z_0} dz$

$\Rightarrow |w(z_0)| \leq \max |w(z)|_{z \in C}$

$\Rightarrow |f(z_0)| \geq \min |f(z)|_{z \in C}$

Given $|f(z)| \geq M$ on C.

$\Rightarrow |f(z_0)| \geq M$ $\forall$ z_0 inside C.

7). $f(z) = f^*(z)$

~~Laurent~~ $f(z) = \sum_{m=0}^{\infty} a_m z^m + \sum_{m=1}^{N} a_{-m} z^{-m}$

$\therefore g(z) = z^N f(z) = \sum_{m=0}^{\infty} a_m z^{m+N} + \sum_{m=1}^{N} a_{-m} z^{-m+N}$

$\cancel{= \sum_{m=0}^{\infty} \hat{a} \, z + \sum_{m=1}^{N} \hat{a}}$ (Next page)

$= \sum$

$$= \sum_{p=N}^{\infty} \hat{a}_p z^p + \sum_{p=0}^{N-1} \hat{a}_p z^p$$

$$= \sum_{p=0}^{\infty} \hat{a}_p z^p .$$

$\therefore$ $g(z)$ is an analytic fu^n every where inside & on a closed contour C ; ~~[illegible]~~ around the origin.

$\because$ $f(z)$ is real, $g(z)$ and all its derivatives is also real $\forall$ real z, in particular at $z = 0$.

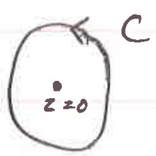
The above series is just a Taylor series of $g(z)$ around $z = z_0$.

$\therefore$ $\hat{a}_p \cong g^p(z_0) \longrightarrow$ real.

8. Was done at class:

$$f(z) = \frac{1}{e^z - 1} = \frac{1}{z\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)}$$ around $z = 0$.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$



$$a_n = \frac{1}{2\pi i} \oint_C \frac{dz}{z^{n+2}\left(1 + \frac{z}{2!} + \cdots\right)}$$

$a_n \neq 0 \ \forall \ n \geq -1$. $\therefore$ a_{-1}, a_0 & a_1 needs to be computed.