PHY 310

Mid Sem - 2017.

1. $f(z) = u + iv$ with $u = 2x(c-y)$

analyticity $\Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

$\Rightarrow \dfrac{\partial v}{\partial y} = 2(c-y)$

$\Rightarrow v = 2cy - y^2 + h(x)$

Ill'ly $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

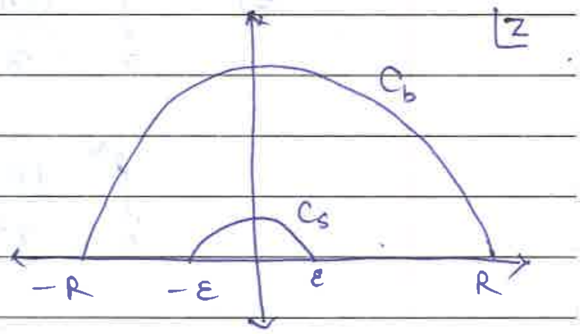
$\Rightarrow -2x = -\dfrac{\partial h}{\partial x} \Rightarrow h = x^2 + C_1$.

$\Rightarrow v = 2cy - y^2 + x^2 + C_1$.

2. a) Assgnmt 3 Q.1, we have proved

$$\int_0^{2\pi} \frac{d\theta}{a - b\cos\theta} = \frac{2\pi}{(a^2-b^2)^{1/2}} \qquad a > |b|.$$

b) Evaluating $\oint_C \dfrac{(\ln z)^2}{z^2+1}\, dz$ on :



Integrand has simple poles at $z = \pm i$, branchpoint at $z = 0$.

So, $\oint_C \dfrac{(\ln z)^2}{z^2+1}\, dz = 2\pi i \operatorname{Res}(i) = \dfrac{-\pi^3}{4}$

$$= \left(\int_{-R}^{-\varepsilon} + \int_{C_s} + \int_{\varepsilon}^{R} + \int_{C_b} \right) \frac{(\ln z)^2}{z^2+1}\, dz$$

Now, $\displaystyle\int_{-R}^{-\varepsilon} \frac{(\ln z)^2\, dz}{z^2+1} = -\int_{\varepsilon}^{R} \frac{(\ln z + i\pi)^2}{z^2+1}\, dz \quad \ldots (z \to -z)$

Also, $\lim\limits_{R\to\infty} \int_{C_r} \frac{(\ln z)^2}{z^2+1}\, dz = 0$.

$\lim\limits_{\varepsilon \to 0} \int_{C_\varepsilon} \frac{(\ln z)^2}{z^2+1}\, dz = 0 \quad \ldots \quad \lim\limits_{x\to 0} x(\ln x)^2$

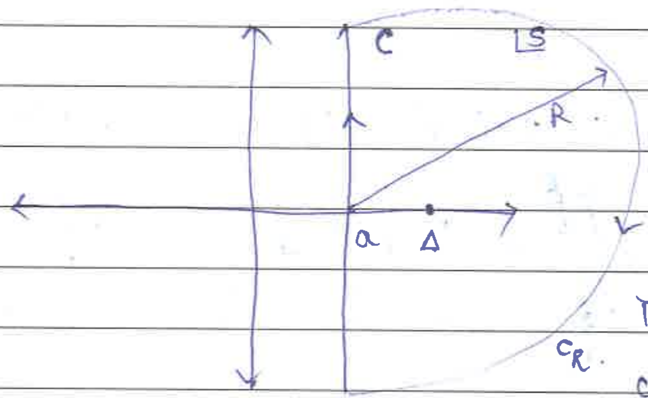
So, $\lim\limits_{\substack{\varepsilon\to 0 \\ R\to\infty}} \int_{\varepsilon}^{R} \frac{(\ln x + i\pi)^2}{x^2+1}\, dx + \int_{\varepsilon}^{R} \frac{(\ln x)^2}{x^2+1}\, dx = \frac{-\pi^3}{4}$.

$$\Rightarrow 2\int_0^\infty \frac{(\ln x)^2}{x^2+1}\, dx + 2\pi i \int_0^\infty \frac{\ln x}{x^2+1}\, dx$$

$$-\pi^2 \underbrace{\int_0^\infty \frac{dx}{x^2+1}}_{=\pi/2} = \frac{-\pi^3}{4}.$$

$$\Rightarrow \int_0^\infty \frac{(\ln x)^2}{x^2+1}\, dx = \frac{\pi^3}{8}.$$

$$\int_0^\infty \frac{\ln x}{x^2+1}\, dx = 0.$$



$$f(z) = \int_{a-i\infty}^{a+i\infty} \underbrace{\frac{z^{-s}}{(s-\Delta)^2}}_{I}\, ds$$

The integrand has branch cut along negative real

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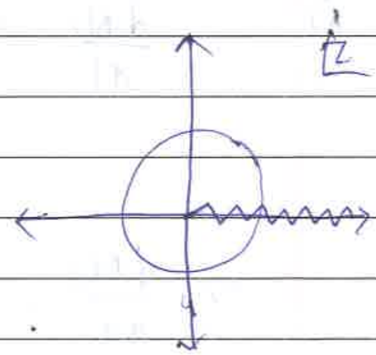
$$= 2\pi i \times \frac{d}{ds} z^{-s} \Big|_{s=\Delta}$$

$$= 2\pi i \times (-\ln z)\, z^{-\Delta}$$

$$\lim_{R\to\infty} \int_{C_R} I\, ds = 0 .$$

$$\Rightarrow f(z) = -2\pi i\, z^{-\Delta} \ln z .$$

Taking a loop around the origin:



we see that the function does not return to its value as we go around the loop.

$$\lim_{\theta\to 0^+} f(ae^{i\theta}) \neq \lim_{\theta\to 0^-\,(2\pi)} f(ae^{i\theta})$$

So, the origin is a branch pt.

let's look at $f(z')$ with $z' = 1/z$.

Again, $z' = 0$ is a branch pt.

So, $z = 0, \infty$ are branch pts and we get the branch cut by joining them.

4 a) we have, $m \dfrac{d^2x}{dt^2} = F(x)$

$$\Rightarrow m \frac{d^2x}{dt^2} \frac{dx}{dt} = F(x) \frac{dx}{dt}$$

$$\Rightarrow \frac{1}{2} m \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = F(x) \frac{dx}{dt}$$

$$\Rightarrow \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 = \int_0^x F(x')\,dx' + C.$$

Now, $v = 0$ at $x = 0$.

$$\Rightarrow \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 = \int_0^x F(x')\,dx'.$$

b) $\frac{dN_1}{dt} = -\lambda_1 N_1 \Rightarrow \frac{dN_1}{N_1} = -\lambda_1 dt$

$$\Rightarrow N_1 = N_0 e^{-\lambda_1 t}$$

$$\Rightarrow \frac{dN_2}{dt} = \lambda_1 N_0 e^{-\lambda_1 t} - \lambda_2 N_2.$$

This is linear first order DE.

Integrating factor $= e^{+\lambda_2 t}$.

$$\Rightarrow N_2(t)\, e^{+\lambda_2 t} = \lambda_1 N_0 \int e^{-\lambda_1 t} e^{+\lambda_2 t}\, dt + C$$

$$= -\lambda_1 N_0 \frac{e^{-(\lambda_1 - \lambda_2)t}}{(\lambda_1 - \lambda_2)} + C.$$

at $t = 0$, $N_2 = 0$.

$$\Rightarrow C = \frac{\lambda_1 N_0}{\lambda_1 - \lambda_2}$$

$$\Rightarrow N_2(t) = e^{-\lambda_2 t} \frac{\lambda_1 N_0}{\lambda_1 - \lambda_2}\left(-e^{+(\lambda_2 - \lambda_1)t} + 1\right)$$

$$= \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}\left(-e^{-\lambda_2 t} + e^{-\lambda_1 t}\right)$$