(1)

[A proof is given in
ARFKEN, WEBER (2005). Mathematical Methods for Physicists; 6e, section 11.2 p 694.
This is an alternate approach.]

The Bessel's ODE

$$x^2 Z_\nu''(x) + x Z_\nu'(x) + (x^2 - \nu^2) Z_\nu(x) = 0$$

$$\Rightarrow \rho^2 \frac{d^2}{d\rho^2} Z_\nu(k\rho) + \rho \frac{d}{d\rho} Z_\nu(k\rho) + (k^2\rho^2 - \nu^2) Z_\nu(k\rho) = 0$$

can be rearranged as

$$-\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{\nu^2}{\rho^2}\right) Z_\nu(k\rho) = k^2 Z_\nu(k\rho)$$

implying the $Z_\nu(k\rho)$ is an eigenfunction of the operator

$$\mathcal{L} = -\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{\nu^2}{\rho^2}\right)$$

with eigenvalue k^2. The weight function needed to make $\mathcal{L}$ self-adjoint is

$$w(\rho) = \frac{1}{p_0(\rho)} e^{\int \frac{p_1(\rho)}{p_0(\rho)} d\rho} = \frac{1}{-1} e^{\int \frac{1}{\rho} d\rho} = -\rho$$

where p_0 and p_1 are coefficients of $\frac{d^2}{d\rho^2}$ and $\frac{d}{d\rho}$ respectively.
Also, if λ_u and λ_v are eigenvalues for eigenfunctions $u(x)$ and $v(x)$ respectively, we have

$$(\lambda_u - \lambda_v)\int_a^b v^* u \, w \, dx = \left[w p_0 (v^* u' - v^{*\prime} u)\right]_a^b$$

where $*$ and $'$ denote conjugation and x-derivatives respectively.

Taking our eigenfunctions to be $J_\nu(k_1\rho)$ and $J_\nu(k_2\rho)$, we get

$$(k_1^2 - k_2^2)\int_0^a (-\rho)\, J_\nu(k_1\rho) J_\nu(k_2\rho)\, d\rho$$

$$= \cancel{a} + \rho \left[J_\nu(k_2 a) .\, k_1 J_\nu'(k_1 a) - k_2 J_\nu'(k_2 a) J_\nu(k_1 a) \right]\Bigg|_{\rho=0}^{\rho=a}$$

$\left[\text{because } \frac{df(x)}{dx} = a f'(xa)\right]$

$$\Rightarrow \quad \frac{a\left[k_2 J_\nu(k_1 a) J'(k_2 a) - k_1 J_\nu'(k_1 a) J_\nu(k_2 a)\right]}{k_1^2 - k_2^2} = \int_0^a \rho\, J_\nu(k_1\rho) J_\nu(k_2\rho)\, d\rho$$

The LHS can be made zero by choosing $k_1 a$ and $k_2 a$ to be zeroes of J_ν, ie, if $\alpha_{\nu i}$ is the i^{th} zero of J_ν, we have the following orthogonality in $[0, a]$:

$$\int_0^a \rho\, J_\nu\left(\alpha_{\nu i}\frac{\rho}{a}\right) J_\nu\left(\alpha_{\nu j}\frac{\rho}{a}\right) d\rho = 0 \quad ; i \neq j$$

Alternatively, taking $\underline{a}$ and $\underline{b}$ as zeroes of J_ν, we can prove the orthogonality as follows:

Rewrite the Bessel's ODE as

$$x\,(x J_\nu')' + (x^2 - \nu^2) J_\nu = 0$$

$$J_\nu(a) = J_\nu(b) = 0$$

Letting $u = J_\nu(ax)$, $v = J_\nu(bx)$ and considering the fact that $x^n f^{(n)}(x) = x^n \frac{d^n}{dx^n} f(x) = x^n \frac{d^n}{d(ax)^n} f(ax) = x^n f^{(n)}(ax)$

we have $J_\nu(ax)$ and $J_\nu(bx)$ satisfying

$$x(xu')' + (a^2x^2 - \nu^2)\, u = 0 \quad \ldots \quad ①$$

$$x(xv')' + (b^2x^2 - \nu^2)\, v = 0 \quad \ldots \quad ②$$

$u.②\ -\ v.①\ \Rightarrow$

$$(b^2 - a^2)\, xuv = \frac{d}{dx}(vxu' - uxv')$$

$$(b^2 - a^2)\int_0^L xuv\, dx = (vxu' - uxv')\Big|_0^L = 0$$

for $v\left(\frac{bx}{L}\right)$, $u\left(\frac{ax}{L}\right)$, hence

$$\int_0^L x\, J_\nu\left(a\frac{x}{L}\right) J_\nu\left(b\frac{x}{L}\right) dx = 0 \quad \text{for } a \neq b$$

② The generating function for J_n is given by

$$g(x, t) = e^{(x/2)(t - t^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

$$\Rightarrow \quad g(u+v, t) = g(u, t) \cdot g(v, t)$$

and

$$\sum_{n=-\infty}^{\infty} J_n(u+v)\, t^n = \sum_{m=-\infty}^{\infty} J_m(u)\, t^m \sum_{l=-\infty}^{\infty} J_l(v)\, t^l$$

Equating the coefficients of t^n, $[l = n - m]$

$$J_n(u+v) = \sum_{m=-\infty}^{\infty} J_m(u)\, J_{n-m}(v)$$

(3) $$g(\mu) = \int_0^1 (1-r^2)\, J_0(\mu r)\, r\,dr$$ [MISWRITTEN IN ASSIGNMENT SHEET]

Changing the integration variables to $x = \mu r$

$$g(x = \mu r) = \int_0^{\mu} \left(1 - \frac{x^2}{\mu^2}\right) J_0(x) \frac{x}{\mu^2}\, dx$$

Integrating by parts,

$$g(x) = \left[\frac{1}{\mu^2}\left(1-\frac{x^2}{\mu^2}\right)\int x J_0(x)\,dx\right]_0^{\mu} - \frac{1}{\mu^2}\int \left(\frac{-2x}{\mu^2}\right)\int x J_0(x)\,dx\; dx$$

$$= \frac{1}{\mu^2}\left[x J_1(x)\left(1-\frac{x^2}{\mu^2}\right)\Bigg|_0^1 + \frac{2}{\mu^2}\int x^2 J_1(x)\,dx \right]$$

[because $[x^n J_n(x)]' = x^n J_{n-1}(x)$]

$$= 0 + \frac{2}{\mu^4}\int_0^{\mu} [x^2 J_2(x)]'\,dx$$

$$= \frac{2}{\mu^4}\; x^2 J_2(x)\Big|_0^{\mu} = \frac{2}{\mu^2} J_2(\mu)$$

(4)

Separating the variables $u = R(\rho)\Phi(\phi)T(t)$, we get the separated equations:

$$\frac{1}{v^2}\frac{d^2T}{dt^2} = -k^2 \Rightarrow T(t) = b_1 e^{i\omega t} + b_2 e^{-i\omega t} \quad \dots (1a)$$

$$\text{where } k^2 = \omega^2/v^2 \quad \dots (1b)$$

$$\frac{d^2\Phi}{d\phi^2} = -m^2 \Rightarrow \Phi(\phi) = a_1 e^{im\phi} + a_2 e^{-im\phi}$$
... (2)

where m is an integer so that ϕ is continuous $\forall \phi$;
... (2b)

The third equation

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \frac{m^2}{\rho^2}R = -k^2 R$$

is a Bessel ODE with variable $k\rho$.

The solution $J_m(k\rho)$ must vanish at $\rho = a$, so that ka are zeroes of J_m. ... (3)

From (1a), (2) and (3),

a) $$u(\rho,\phi,t) = J_m(k\rho)\left[a_1 e^{im\phi} + a_2 e^{-im\phi}\right]\left[b_1 e^{i\omega t} + b_2 e^{-i\omega t}\right]$$

From (1b) and (3),

b) The allowed values of k are

$$k_n = \alpha_{mn}/a = \omega_n/v$$

$m \in \mathbb{Z}$

where α_{mn} is the n^{th} zero of J_m. Hence both k and ω are discrete.