ASG 7 Sol :

1. $\because$ eqn & boundary condn are invariant under $(x,t) \to (sx, s^2 t)$

$\therefore$ soln is also invariant.

$\therefore \quad u(x,t) = u(sx, s^2 t)$.

For $t>0$, put $s = \frac{1}{\sqrt{t}}$. $\therefore \; u(x,t) = u\left(\frac{x}{\sqrt{t}}, 1\right)$; fun^n of single variable $\left(\frac{x}{\sqrt{t}}\right) \equiv y$.

Writing the eqn interms of y,

$$\frac{d^2 u}{dy^2} + \frac{y}{2a^2}\frac{du}{dy} = 0$$ and boundary cond^s are.

$$u(-\infty) = 0, \quad u(\infty) = 1.$$

Integrating we get,

$$u(y) = \text{Const} \int_{-\infty}^{y} e^{-\frac{y'^2}{4a^2}} \, dy'$$

$$\therefore \quad u(x,t) = \frac{1}{2a\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{t}} e^{-\frac{y'^2}{4a^2}} \, dy'.$$

No soln for $t<0$. We have diffusion eqn ; hence we can't have a situation where a system where a system that has disorganized infinite amount of times becomes organized at $t=0$.

2. In cylindrical coordinates ; Helmholtz eqn looks like,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0.$$

Replace $k^2 \to k'^2 + f(\rho) + \frac{1}{\rho^2} g(\phi) + h(z)$.

and $\psi(\rho, \phi, z) = R(\rho)\, \Phi(\phi)\, Z(z)$,

The separated eqn's are,

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + h(z) = A^2 - k'^2 \qquad \boxed{A = \text{cont}}$$

$$\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} + g(\phi) + m^2 = 0, \qquad \boxed{m = \text{cont}}$$

$$\frac{\rho}{R}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R}{\partial \rho}\right) + \rho^2 f(\rho) + A^2 \rho^2 - m^2 = 0.$$

These are three eqn's that only involve one variable each.

Assignment 8 soln

1. $(\nabla^2 + k^2)\, G(r, r') = \delta(r - r')$ — ①

Solving the eqn in Fourier space,

$$G \equiv \int \frac{d^3p}{(2\pi)^3}\, g_0(p)\, e^{ip\cdot(r-r')}$$

$$① \Rightarrow g_0(p) = \frac{-1}{k^2 - p^2}$$

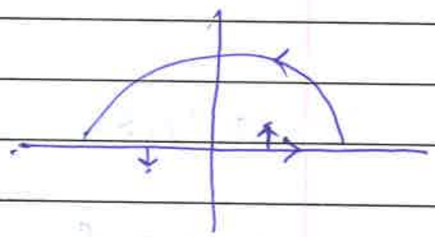
let $p\cdot(r-r') = p|r-r'|\cos\theta$.

$$\Rightarrow G(r,r') = \frac{2\pi}{(2\pi)^3}\int \frac{p^2\,dp}{p^2-k^2} \int d(\cos\theta)\, e^{ip|r-r'|\cos\theta}$$

$$= \frac{1}{(4\pi)^2}\,\frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ip|r-r'|} - e^{-ip|r-r'|}}{(p^2-k^2)\times i|r-r'|}\, p\,dp$$

Performing the integrals by complexifying them.
(shift the poles by ϵ & set ϵ to 0)

$$\oint \frac{e^{ip|r-r'|}}{p^2-k^2}\, p\,dp$$



poles : $p = \pm k \pm i\epsilon$

$$I_1 = \frac{e^{ik|r-r'|}}{8\pi|r-r'|}$$

finally $G(r,r') = \dfrac{e^{ik|r-r'|}}{4\pi|r-r'|}$

2. Charged conducting ring : $\rho(\vec{r}') = \dfrac{q}{2\pi a^2}\,\delta(r'-a)\,\delta(\cos\theta')$

As, the charge is finite we can use the usual b.c. of $\phi \to 0$ at infinity.
(Note that in the class we solved the problem for $\phi = 0$ for $r = a$).

~~$a \leq r < \infty$~~ For $\phi = 0$ at $r = \infty$, the Green's function is :

$$G(r, r') = \frac{1}{|r - r'|}$$

$$\phi(\vec{r}) = \int \frac{1}{|r - r'|}\,\rho(r')\,r'^2 dr' \sin\theta\, d\theta\, d\phi$$

$$|r - r'|^2 = r^2 + r'^2 - 2rr'\cos(\theta - \theta')$$

So, $\phi(\vec{r}) = \dfrac{\cancel{2\pi a^2} q}{|r^2 + a^2 - 2ra\sin\theta|^{1/2}}$ for $0 \leq r < \infty$

~~For, $0 \leq r < a$, the equation is $\nabla^2\phi = 0$.~~
~~with $\phi(a, \theta) = \dfrac{q}{\sqrt{2}\,a\,|1 - \sin\theta|^{1/2}}$~~

~~$\phi(r, \theta) = \sum A_l r^l P_l(\cos\theta)$~~

~~$\Rightarrow A_l = \dfrac{q}{\sqrt{2}\,a^{l+1}} \underbrace{\int_{-1}^{1} (1-x)^{-1/2} P_l(x)\,dx}_{c_l}$~~

~~$c_0 = 2\sqrt{2}$~~
~~$c_1 = \dfrac{2\sqrt{2}}{3}$~~
~~$c_2 = -2\sqrt{2}/15$ ....~~

3. In one of the classes, the free space Green function for wave equation has been derived.

$$G(\vec{r}, t; \vec{r}', t') = \frac{\delta(t' - t + |\vec{r} - \vec{r}'|)}{(-4\pi)\,|\vec{r} - \vec{r}'|}$$

The solution satisfies homogenous b.c. :

$$\phi ;\ \partial_r \phi \to 0 \quad \text{as } r \to \infty.$$

Now, $G(\vec{r},t;\vec{r}',t') \to 0$, $\partial_r G(\vec{r},t;\vec{r}',t') \to 0$ as $r \to \infty$.

So, required solution :

$$\phi(r,t) = \int r'^2 dr' \sin\theta' d\theta' d\phi' \frac{\delta(t'-t+|\vec{r}-\vec{r}'|)}{-4\pi|\vec{r}-\vec{r}'|} \rho(\vec{r}',t')\,dt'$$

$$= \frac{1}{-4\pi\epsilon_0}\int r' dr' \sin\theta' d\theta' d\phi' \frac{\rho(\vec{r}',t \pm |\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}$$

This is the Lienard-Wiechert potential.

We can clearly see how the effect of source at previous instant of time affects potential at given time.

Quiz 2 Solution of Q.2

For, $y'' + Py' + Qy = 0$, $W(x) = W_0 e^{-\int^x P(x')dx'}$;

(See, for e.g. pg 455 Arfken edition reprint 2010).

Given $W(x_0) = 0$.

~~The~~ Exponential function cannot vanish. ($\int P' dx'$ does not diverge).

Hence, $W_0 = 0$.

Hence, $W(x) = 0$ ~~fall~~ for all points.

$\Rightarrow y_1 y_2' - y_1' y_2 = 0 \ \forall x$.

$\Rightarrow y_1 = c_1 y_2 \ \forall x$, hence, linearly dependent.