Assignment 5 - soln.

1. G.S. of homogenous eqn is $y_h = c_1 \sin\omega x + c_2 \cos\omega x$

For $x \neq \xi$, $\nabla_x^2 G(x,\xi) = 0$

$$G(x,\xi) = c_1(\xi) \sin\omega x + c_2(\xi) \cos\omega x \quad \ldots x < \xi$$
$$= d_1(\xi) \cos\omega x + d_2(\xi) \sin\omega x \quad \ldots x > \xi$$

b.c. $\Rightarrow$ $G(0,\xi) = 0 \Rightarrow c_2 = 0$

$G(1,\xi) = 0 \Rightarrow d_1 = -d_2 \tan\omega$ — ①

Continuity of $G(x,\xi)$ at $x = \xi$:

$$c_1 \cos\omega\xi + c_2 \sin\omega\xi = d_1 \cos\omega\xi + d_2 \sin\omega\xi \quad \text{— ②}$$

Discty of derivative of G at $x = \xi$:

$$-\omega c_2 \cos\omega\xi - \omega d_1 \sin\omega\xi + d_2 \omega \cos\omega\xi = 1 \quad \text{— ③}$$

Solve ①, ②, ③ for c_2, d_1, d_2.

Finally,

$$G(x,\xi) = \frac{\sin\omega\xi}{\omega \sin\omega} \sin(\omega x - \omega) \quad \ldots x > \xi$$

$$= \frac{1}{\omega \sin\omega} \sin(\omega\xi - \omega)\, \omega \sin x \quad \ldots x < \xi$$

2. $\mathcal{L}[y] = l(x) y'' + p(x) y' + q(x) y$

$\mathcal{L}^*[y] = l y'' + (2l' - p) y' + (l'' - p' + q) y$ — ①

4

with $\langle v|u\rangle = \int v^*(x)\, u(x)\, dx$.

Eqn ① can be obtained by definition of adjoint

$\langle v|\mathcal{L}u\rangle = \langle \mathcal{L}^* v|u\rangle$ using above definition of inner product.
(use integration by parts).

For operator $h\mathcal{L}$, $l_1 = hl$.
$p_1 = hp$
$q_1 = qh$ ~~using notation of~~

$(h\mathcal{L})^{+}[y] = l_1 y'' + (2l_1' - p_1)\, y' + (l_1'' - p_1' + q_1) z = 0$

now, $l_1' = h'l + l'h$
$h' = -\frac{l'}{l} h + h$

$\Rightarrow l_1' = hp = p_1$.

So, $(h\mathcal{L})^{+}[y] = hly'' + hp\, y' + hqy$
$= h\mathcal{L}$.
hence, $h\mathcal{L}$ is self adjoint.

3. Substituting $y = ve^x$ in D.E.

$x(v'e^x + ve^x)' - (2x+1)(v'e^x + ve^x) + (x+2)ve^x = 0$

$\Rightarrow v\left[xe^{x} - 2(x+1)e^{x} + (x+2)e^{x}\right]$

$+ xv'' + 2v'xe^{x} - 2(x+1)v'e^{x} = 0$

The term in [ ] is $\mathcal{L}_x e^{x}$ which is 0 as e^{x} is soln to $\mathcal{L}_x y(x) = 0$

So, $xv'' + 2v'\cancel{x}e^{x} - 2(\cancel{x}+1)v'\cancel{e^x} = 0$

This is first order differential eqn for $v'(x)$

$$x\frac{dv'}{dx} = 2v'$$

$$\Rightarrow v(x) = 2\int^{x} \ln x'\,dx' + c$$

4. Refer to Sect 6 of Chp 8 of Math Method book by Mary Boas 3rd edition for general discussion.

For source $12xe^{3x}$ & homogenous eqn $(\partial_y - 3)^2 y = 0$,

we assume an ansatz $y = Ax^3e^{3x}$.

$y' = \left[(3x^2 + 3x^3)e^{3x}\right]$

$y'' = \left[(6x + 9x^2)e^{3x} + 9(x^2+x^3)e^{3x}\right]$

Comparing the powers on two sides, we get $A = 2$ due to the term xe^{x} other powers viz x^2e^{x} & x^3e^{x} cancel due to choice of ansatz.