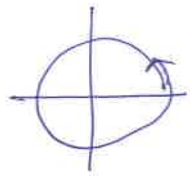
<u>Assignment 3</u> - solutions.

① $I = \int_0^{2\pi} \frac{d\theta}{a - b\cos\theta}$ $\quad |a| > |b|$.

write this as a complex integral over contour of unit circle.

hence, $z = e^{i\theta} \Rightarrow \cos\theta = \frac{z + 1/z}{2}$.

$$\Rightarrow I = \oint \frac{-i(e^{-i\theta}dz)}{a - b\left(\frac{z^2+1}{2z}\right)}.$$



$$= -i \oint \frac{2\,dz}{2za - bz^2 - b}.$$

Use Residue theorem to evaluate the integral.

- There are simple poles. check which lie inside the contour for given condition.

② $I = \int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2}$

Now, using change of variables $\theta \to 2\pi - \theta$, we proved,

$$I = \int_{\pi}^{2\pi} \frac{d\theta}{(a+\cos\theta)^2}.$$

So, $I = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2}$

Again, writing it as complex integral over unit circle. with $z = e^{i\theta}$.

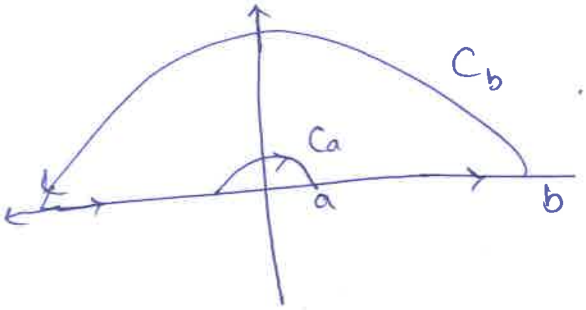
$$I = -\frac{i}{2}\oint \frac{z\,dz}{\frac{(z^2+1+2za)^2}{4}}.$$

use Residue theorem. (Now there are double poles)

③ $I = \int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx .$

Evaluating $\oint_C \frac{1 - e^{2iz}}{z^2}\, dz$

C :



using residue theorem for contour C.

$$\left[\int_{-b}^{-a} + \int_{C_a} + \int_a^b + \int_{C_b} \right] \underbrace{\frac{1-e^{2iz}}{z^2}}_{f(z)} dz = 0 .$$

Now $\int_{C_b} f(z)\, dz = 0$ as $b \to \infty$.

$$\lim_{\substack{a \to 0 \\ b \to \infty}} \left[\int_{-b}^{-a} + \int_a^b \right] \frac{1-e^{2iz}}{z^2}\, dz = 4I .$$

$$\Rightarrow 4I = \lim_{a \to 0} - \int_{C_a} \frac{1-e^{2iz}}{z^2}\, dz$$

$$= \lim_{a \to 0} \int_{C_a} \frac{\left(2iz + \frac{(2iz)^2}{2!} + \cdots\right)}{z^2}\, dz .$$

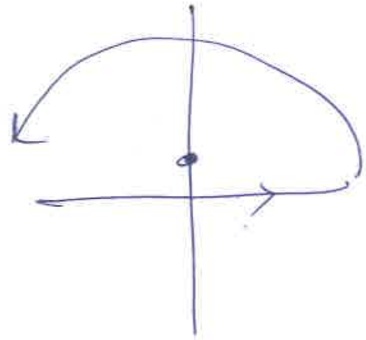
$$\mathrm{Re}\left[\frac{1-e^{2iz}}{2}\right] = \sin^2 z$$

$$= \lim_{a \to 0} \int_{\pi}^{0} \frac{(2iae^{i\theta} + \cdots)\, ae^{i\theta}\, i\, d\theta}{a^2 e^{2i\theta}}$$

$$= 2\pi . \qquad \Rightarrow I = \pi/2 .$$

Q. 2 a) $I = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{izs}}{z - i\epsilon}$

S > 0.



simple pole at $z = i\epsilon$

For $s > 0$, closing the contour upwards,

$$I = 2\pi i \times \frac{1}{2\pi i} e^{-\epsilon s}$$

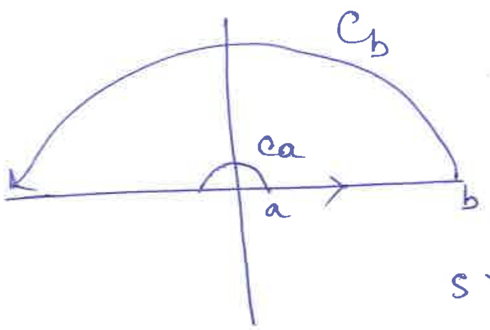
$$= 1 \quad \text{as } \epsilon \to 0.$$

For $s < 0$, closing downwards,

$$\Rightarrow I = 0.$$

$$\Rightarrow \Theta(s) = \lim_{\epsilon \to 0} I.$$

b) using the contour : (for $s > 0$).



s > 0.

$$I = \oint_C \frac{e^{izs}}{z} dz = 0.$$

using $\int_{C_b} \frac{e^{izs}}{z} dz = 0$ as $b \to \infty$,

$$P \int_{-\infty}^{\infty} \frac{e^{ixs}}{x} dx = -\int_{C_a} \frac{e^{izs}}{z} dz.$$

$$= i\pi \text{ Residue } (z = 0)$$

$$= i\pi.$$

For $s < 0$, closing downwards,

$$P \int_{-\infty}^{\infty} \frac{e^{ixs}}{x} \, dx + \int_{C_S} \frac{e^{izs}}{z} \, dz = -2\pi i \, \mathrm{Resi}(z=0)$$

$$\Rightarrow \quad P \int_{-\infty}^{\infty} \frac{e^{ixs}}{x} \, dx = -2\pi i + i\pi$$

$$\Rightarrow \quad \frac{1}{2} + \frac{P}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{ixs}}{x} \, dx = \frac{1}{2} + \frac{1}{2} = 1 \qquad s > 0$$

$$= \frac{1}{2} - \frac{1}{2} = 0 \qquad s < 0$$