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Quiz - 2

1. $\langle x \rangle = \int \psi^* x \psi \, d^3x$

$\langle p_x \rangle = \int \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi \, d^3x$

$$\frac{d}{dt}\langle x \rangle = \int \left(\frac{\partial \psi^*}{\partial t} x \psi + \psi^* x \frac{\partial \psi}{\partial t} \right) d^3x$$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi$$ [Take V real].

$$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi^* + V\psi^*$$

$$\therefore \frac{d\langle x \rangle}{dt} = \int \left[\frac{i}{\hbar} \left[-\frac{\hbar^2}{2m} \nabla^2 \psi^* + V\psi^* \right] x\psi - \frac{i}{\hbar}\psi^* x \left[-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi \right] \right] d^3x$$

The V term cancels.

$$\therefore \frac{d\langle x \rangle}{dt} = \frac{i}{\hbar}\frac{\hbar^2}{2m} \int \left[\psi^* x \nabla^2 \psi - \psi x \nabla^2 \psi^* \right] d^3x$$

Now, $\int \psi x (\nabla^2 \psi^*) \, d^3x$

$$= \underbrace{\int \psi x (\nabla \psi^*) \cdot dS}_{\to 0} - \int \nabla(\psi x) \cdot \nabla(\psi^*) \, d^3x$$

$\because \psi \to 0$ faster than x in the boundary.

$$= -\underbrace{\int \psi^* \nabla(\psi x) \cdot dS}_{\to 0} + \int \psi^* \nabla^2(\psi x) \, d^3x$$

as $\psi^* \to 0$ faster than x in the boundary.

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$$\therefore \int \psi x \nabla^2(\psi^*)\, d^3x$$ $\cancel{\left(\frac{\partial^2}{\partial x^2} + \cdots\right)(\psi x) = \partial\left(\frac{\partial\psi}{\partial x}x + \psi\right) = \left(\nabla^2\psi \cdot x + 2\nabla\psi\right)}$

$$= \int \psi^* \nabla^2(\psi x)\, d^3x.$$

$\therefore$ we get,

$$\frac{d\langle x\rangle}{dt} = \frac{i}{\hbar}\frac{\hbar^2}{2m}\int\left[\psi^* x \nabla^2\psi - \psi^* \nabla^2(\psi x)\right] d^3x$$

$$= \frac{i}{\hbar}\frac{\hbar^2}{2m}\int\left[-2\psi^*\frac{\partial\psi}{\partial x}\, d^3x\right]$$

$$= \frac{1}{m}\int \psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\psi\, d^3x$$

$$= \frac{\langle p_x\rangle}{m}$$

Details

$$\nabla^2(\psi x)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\psi x)$$

$$= \frac{\partial^2}{\partial x^2}(\psi x) + x\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\cdot x + \psi\right) + x\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi$$

$$= x\frac{\partial^2\psi}{\partial x^2} + 2\frac{\partial\psi}{\partial x} + x\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi$$

$$= x\nabla^2\psi + 2\frac{\partial\psi}{\partial x}$$