Solution to MidSem Paper.

1. Given the question already talks about free particle, we can use the formula's derived in class. The form of the wavefunction is unnecessary.

For free particle $V(x,t) = 0$

We know,
$$\frac{d}{dt}\langle p_x\rangle = -\left\langle \frac{dV}{dx}\right\rangle \quad \text{— (1)}$$
$$\& \quad \frac{d}{dt}\langle x\rangle = \frac{\langle p_x\rangle}{m} \quad \text{— (2)}$$

From (1), in case of free particle, $\frac{d\langle p_x\rangle}{dt} = 0.$ — (3)

$\Rightarrow \langle p_x\rangle$ is independent of time.

From (2) & (3) then we get,
$$\langle x\rangle = \frac{\langle p_x\rangle}{m}t + \langle x\rangle_0$$
| where $\langle x\rangle_0$ is const.

Alternative derivation to 1:

$$\psi(x,t) = \frac{1}{(2\pi\hbar)^{1/2}}\int_{-\infty}^{\infty} dp_x \, e^{\frac{i}{\hbar}\left(xp_x - \frac{p_x^2 t}{2m}\right)}\Phi(p_x)$$

$$\therefore \langle p_x\rangle = \int_{-\infty}^{\infty}\psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x,t)\,dx$$
$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp_{1x}\,dp_{2x}\, e^{-\frac{i}{\hbar}\left(xp_{1x} - \frac{p_{1x}^2 t}{2m}\right)}\left(-i\hbar\frac{\partial}{\partial x}\right)e^{\frac{i}{\hbar}\left(xp_{2x} - \frac{p_{2x}^2 t}{2m}\right)}\Phi^*(p_{1x})\,\Phi(p_{2x})\,dx \quad \text{(A)}$$

Differentiating w.r.t. x we get,
$$= \frac{-1}{2\pi\hbar}\int_{-\infty}^{\infty} dp_{1x}\,dp_{2x}\, e^{-\frac{i}{\hbar}\left(xp_{1x} - \frac{p_{1x}^2 t}{2m}\right)}(-i\hbar)\frac{i}{\hbar}p_{2x}\, e^{\frac{i}{\hbar}\left(xp_{2x} - \frac{p_{2x}^2 t}{2m}\right)}\Phi^*(p_{1x})\,\Phi(p_{2x})\,dx$$

Now, we perform the x integral after writing down all the x dependent pieces together.

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dP_{1x}\, dP_{2x}\, e^{-\frac{i}{\hbar}\left(\frac{P_{2x}^2 t}{2m} - \frac{P_{1x}^2 t}{2m}\right)} P_{2x}\, \Phi^*(P_{1x})\, \Phi(P_{2x})\, \underbrace{\int_{-\infty}^{\infty} dx\, e^{-\frac{i}{\hbar}x(P_{1x}-P_{2x})}}$$

$$= 2\pi\, \delta\left(\frac{1}{\hbar}(P_{1x}-P_{2x})\right)$$

$$= 2\pi\hbar\, \delta(P_{1x}-P_{2x})$$

$$= \int_{-\infty}^{\infty} dP_{1x}\, dP_{2x}\, e^{-\frac{i}{\hbar}\left(\frac{P_{2x}^2 t}{2m} - \frac{P_{1x}^2 t}{2m}\right)} P_{2x}\, \delta(P_{1x}-P_{2x})\, \Phi^*(P_{1x})\, \Phi(P_{2x}).$$

Integrating with P_{2x} we get,

$$= \int_{-\infty}^{\infty} dP_{1x}\, P_{1x}\, \Phi^*(P_{1x})\, \Phi(P_{1x})$$

The last expression is independent of time.

$$\therefore \quad \boxed{\frac{d}{dt}\langle P_x\rangle = 0.}$$

$$\text{Now, } \langle x\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\, x\, \psi(x,t)\, dx.$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dP_{1x}\, dP_{2x}\, e^{-\frac{i}{\hbar}\left(P_{1x}x - \frac{P_{1x}^2 t}{2m}\right)}\, x\, e^{\frac{i}{\hbar}\left(P_{2x}x - \frac{P_{2x}^2 t}{2m}\right)}\, \Phi^*(P_{1x})\, \Phi(P_{2x})\, dx \qquad \text{(B)}$$

Now, notice, $\displaystyle \frac{\hbar}{i}\frac{\partial}{\partial P_{2x}} e^{\frac{i}{\hbar}\left(P_{2x}x - \frac{P_{2x}^2 t}{2m}\right)}$

$$= \left(x - \frac{P_{2x}t}{m}\right) e^{\frac{i}{\hbar}\left(P_{2x}x - \frac{P_{2x}^2 t}{2m}\right)}$$

$$\therefore\ x\, e^{\frac{i}{\hbar}\left(P_{2x}x - \frac{P_{2x}^2 t}{2m}\right)} = \frac{\hbar}{i}\frac{\partial}{\partial P_{2x}} e^{\frac{i}{\hbar}\left(P_{2x}x - \frac{P_{2x}^2 t}{2m}\right)} + \frac{P_{2x}t}{m}\, e^{i/\hbar\left(P_{2x}x - \frac{P_{2x}^2 t}{2m}\right)}.$$

Substituting this identity to R.H.S. we get,

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dP_{1x}\, dP_{2x}\; e^{-\frac{i}{\hbar}\left(P_{1x}x - \frac{P_{1x}^2 t}{2m}\right)} \frac{\hbar}{i}\frac{\partial}{\partial P_{2x}} e^{\frac{i}{\hbar}\left(P_{2x}x - \frac{P_{2x}^2 t}{2m}\right)} \Phi^*(P_{1x})\,\Phi(P_{2x})$$

$$+ \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dP_{1x}\, dP_{2x}\; e^{-\frac{i}{\hbar}\left(P_{1x}x - \frac{P_{1x}^2 t}{2m}\right)} \frac{P_{2x}t}{m}\, e^{\frac{i}{\hbar}\left(P_{2x}x - \frac{P_{2x}^2 t}{2m}\right)} \Phi^*(P_{1x})\,\Phi(P_{2x}).$$

$$= I_1 + I_2$$

$I_2 = \frac{t}{m}\langle P_x\rangle$ from (A)

$$I_1 = \frac{1}{2\pi\hbar}\int dP_{1x}\, dP_{2x}\, dx\; \Phi^*(P_{1x}).\left(i\hbar\frac{\partial}{\partial P_{2x}}\right) e^{-\frac{i}{\hbar}(P_{1x}-P_{2x})x}\, e^{\frac{it}{m\hbar}(P_{1x}^2 - P_{2x}^2)}\,\Phi(P_{2x})$$

+ boundary term. (this vanishes at infinity).

Now performing the x integral,

$$= \frac{1}{2\pi\hbar}\int dP_{1x}\, dP_{2x}\; \Phi^*(P_{1x})\left(i\hbar\frac{\partial}{\partial P_{2x}}\right) 2\pi\hbar\,\delta(P_{1x}-P_{2x})\cdot e^{\frac{it}{m\hbar}(P_{1x}^2 - P_{2x}^2)}\,\Phi(P_{2x})$$

Again doing the P_{1x} integral,

$$= \int dP_{2x}\; \Phi^*(P_{2x})\left(i\hbar\frac{\partial}{\partial P_{2x}}\right) \cancel{e^{\frac{it}{m\hbar}}}\;\Phi(P_{2x})$$

This is just $\langle x\rangle$ at $t=0$.

$\therefore\ I_1 = \langle x\rangle_0$

$$\therefore\ \boxed{\langle x\rangle = \langle x\rangle_0 + \frac{t}{m}\langle P_x\rangle}$$

2. (a) $$\Phi(P_x) = \frac{1}{\sqrt{2\alpha}} \quad ; \quad P_0-\alpha \le P_x \le P_0+\alpha.$$

$$\therefore \int_{-\infty}^{\infty} \Phi^*(P_x)\,\Phi(P_x)\,dP_x$$

$$= \frac{1}{2\alpha}\int_{P_0-\alpha}^{P_0+\alpha} dP_x = \frac{1}{2\alpha}\,2\alpha = 1.$$

$\therefore$ ~~The~~ $\Phi(P_x)$ is unit normalised.

(b) The corresponding wavefunction is given by,

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}P_x\cdot x}\,\Phi(P_x)\,dP_x$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\int_{P_0-\alpha}^{P_0+\alpha} \frac{1}{\sqrt{2\alpha}}\,e^{\frac{i}{\hbar}P_x\cdot x}\,dP_x$$

$$= \frac{1}{\sqrt{4\pi\hbar\alpha}}\left.\frac{e^{i/\hbar\,P_x\cdot x}}{i/\hbar\,x}\right|_{P_0-\alpha}^{P_0+\alpha}$$

$$= \frac{-i\hbar}{\sqrt{4\pi\hbar\alpha}}\,e^{\frac{i}{\hbar}P_0x}\,\frac{2i\sin\left(\frac{\alpha x}{\hbar}\right)}{x}$$

$$= \sqrt{\frac{\hbar}{\pi\alpha}}\,e^{\frac{iP_0x}{\hbar}}\,\frac{\sin\left(\frac{\alpha x}{\hbar}\right)}{x}.$$

3.

(i) The potential $V(x)$ has a finite jump at $x = x_0$

Then, (a) $\psi(x)$ & $\psi'(x)$ both will be continuous.

(b) $\psi''(x)$ will have a finite jump.

(c) $\psi(x) \to 0$ as $x \to \pm\infty$.

(ii) If the potential has infinite discontinuity at $x = x_0$ then,

(a) $\psi(x)$ is continuous at $x = x_0$

(b) $\psi'(x)$ is discontinuous at $x = x_0$

(c) $\psi(x) \to 0$ as $x \to \pm\infty$.

(iii) For time independent potential, we may write,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = (E - V(x))\psi.$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = [E + \alpha\,\delta(x)]\,\psi(x) \qquad [\text{A.T.Q.}]$$

$$\Rightarrow -\frac{\hbar^2}{2m}\int_{0_-}^{0_+} d\left(\frac{d\psi}{dx}\right) = E\int_{0_-}^{0_+}\psi(x)\,dx + \alpha\int_{0_-}^{0_+}\delta(x)\,\psi(x)\,dx$$

$$\Rightarrow \left.\frac{d\psi}{dx}\right|_{0_+} - \left.\frac{d\psi}{dx}\right|_{0_-} = \alpha\,\psi(0)\left(-\frac{2m}{\hbar^2}\right).$$

The first integral in RHS vanishes because $\psi(x)$ is continuous.

$$\Rightarrow \boxed{\left.\frac{d\psi}{dx}\right|_{0_+} - \left.\frac{d\psi}{dx}\right|_{0_-} = -\frac{2m\alpha}{\hbar^2}\psi(0).} \qquad (1)$$

(b) Now consider $E<0$.

For $x>0$, $V(x)=0$,

Thus Schrödinger equation becomes;

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = -|E|\psi(x).$$

$$\Rightarrow \frac{d^2\psi(x)}{dx^2} = \frac{2m|E|}{\hbar^2}\psi(x).$$

$$\therefore \psi(x) = Be^{-kx} \quad \text{—— (i)} \quad \text{where } k=\sqrt{\frac{2m|E|}{\hbar^2}}$$

Note that the other solution of the differential equation is absent. Since as $x\to\infty$ $e^{kx}\to\infty$ which is not acceptable for a wavefunction.

Similarly $x<0$,

$$\psi(x) = Ce^{kx} \quad \text{—— (ii)}$$

| Again the solution $\propto e^{-kx}$ is ~~absent~~ not physical as $x\to-\infty$ $\propto e^{-kx}\to\infty$.

Now, $\psi(x)$ must be continuous at $x=0$.

$$\Rightarrow B=C.$$

Using these expressions in (1) we get,

$$\cancel{B}(-k) - \cancel{B}k = -\frac{2m\alpha}{\hbar^2}\cancel{B}$$

$$\Rightarrow -\cancel{2}k = -\frac{\cancel{2}m\alpha}{\hbar^2} \quad \Rightarrow k^2 = \frac{m^2\alpha^2}{\hbar^4}$$

$$\Rightarrow \frac{2m|E|}{\hbar^2} = \frac{m^2\alpha^2}{\hbar^4}$$

$$\Rightarrow \boxed{|E| = \frac{\alpha^2 m}{2\hbar^2}}$$

(c) The complete wavefunction for this problem is then,

$$\psi(x,t) = \psi(x)e^{-iEt/\hbar} = Be^{\sqrt{\frac{2m|E|}{\hbar^2}}|x|}e^{-iEt/\hbar}$$

$$\& \quad E = -\frac{m\alpha^2}{2\hbar^2} \quad \text{[Ans]}.$$