PHY202: Assignment 3 solutions

1.

$$\begin{aligned}
\langle X\rangle &= \int_{-\infty}^{\infty}\psi^*(x)\,x\psi(x)\,dx\\
&= \int\int\int dp\,dp'\phi^*(p')\,e^{\frac{ip'x}{\hbar}}xe^{-\frac{ipx}{\hbar}}\phi(p)\,dx \qquad \textit{Expanding } \psi \textit{ and } \psi^* \textit{ in momentum basis}\\
&= \int\int dp'\phi^*(p')\,e^{\frac{ip'x}{\hbar}}\int dp\phi(p)\,i\hbar\frac{\partial}{\partial p}e^{-\frac{ipx}{\hbar}}\,dx
\end{aligned}$$

Where we've used the identity $xe^{-\frac{ipx}{\hbar}} = i\hbar\frac{\partial}{\partial p}e^{-\frac{ipx}{\hbar}}$

$$\begin{aligned}
&= \int\int dp'\phi^*(p')\,e^{\frac{ip'x}{\hbar}}\int dpe^{-\frac{ipx}{\hbar}}i\hbar\frac{\partial}{\partial p}\phi(p)\,dx \quad \textit{Integrating by parts}\\
&= \int\int\int dp\,dp'\phi^*(p')\,e^{\frac{i(p'-p)x}{\hbar}}i\hbar\frac{\partial}{\partial p}\phi(p)\,dx\\
&= \int\int dp\,dp'\phi^*(p')\,i\hbar\frac{\partial}{\partial p}\phi(p)\,\delta(p'-p) \qquad \textit{Performing the } x \textit{ integral}\\
&= \int dp\phi^*(p)\,i\hbar\frac{\partial}{\partial p}\phi(p) \qquad \textit{Performing } p' \textit{ integral}
\end{aligned}$$

Now everything is in momentum basis. Hence we conclude $\hat{X} = i\hbar\frac{\partial}{\partial p}$ in momentum basis.

2. We know already that

$$\begin{aligned}
\langle p\rangle &= \int_{-\infty}^{\infty}\psi^*(-i\hbar)\,\partial_x\psi\\
\implies \frac{d\langle p\rangle}{dt} &= -i\hbar\int_{-\infty}^{\infty}\left(\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial x} + \psi^*\frac{\partial^2\psi}{\partial x^2}\right)dx\\
&= \int_{-\infty}^{\infty}\left[\left(-i\hbar\frac{\partial\psi^*}{\partial t}\right)\frac{\partial\psi}{\partial x} + \frac{\partial\psi^*}{\partial x}\left(i\hbar\frac{\partial\psi}{\partial t}\right)\right]dx
\end{aligned}$$

Now using Schrodinger's equation and it's complex conjugate we can expand the two terms above to get,

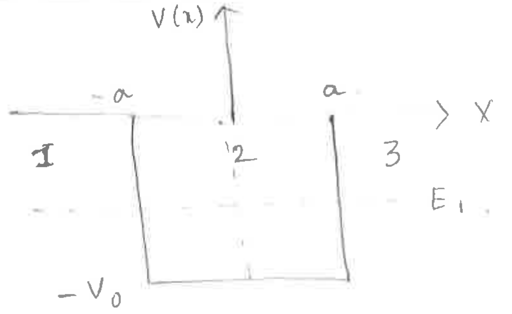
$$\frac{d\langle p\rangle}{dt} = \int_{-\infty}^{\infty}\left[\frac{-\hbar^2}{2m}\frac{\partial}{\partial x}\left(\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}\right) + V(x)\,\frac{\partial|\psi^2|}{\partial x}\right]dx$$

The first term is a total derivative and thus will give zero after integration and putting limits. The second term can be integrated by parts to give,

$$\int_{-\infty}^{\infty}V(x)\,\frac{\partial|\psi^2|}{\partial x}dx = \int_{-\infty}^{\infty}\frac{dV}{dx}|\psi^2|dx = \langle\frac{dV}{dx}\rangle$$

$$Thus, \frac{d\langle p\rangle}{dt} = \langle\frac{dV}{dx}\rangle \qquad [Q.E.D]$$

3.



region 1 ; $x < -a$

region 2 ; $-a < x < a$

region 3 ; $x > a$

$$\left[\tilde{A} e^{ikx} + \tilde{B} e^{-ikx} \quad A = 2(\tilde{A} + \tilde{B}), \; B = 2i(\tilde{A} - \tilde{B}) \right]$$

Sol^n ; in reg 2 $\quad \psi(x) = A \cos Kx + B \sin Kx . \quad |x| < a$

$$K = \frac{1}{\hbar}\sqrt{2m(V_0 - E_1)}$$

In reg 1 : $\psi(x) = C e^{Lx} + D e^{-Lx} \quad x < -a$

In reg 3 : $\psi(x) = D' e^{Lx} + C' e^{-Lx} \quad x > a$

$$L = \frac{1}{\hbar}\sqrt{2mE_1} .$$

1) • Now, $\psi(x)$ & $\frac{d\psi}{dx}$ has to be continuous at $x = \pm a$.

2) • And $\psi(x)$ has to be finite over whole space.

2) $\Rightarrow \quad D = D' = 0$.

1) $\Rightarrow$

$$C e^{-La} = A \cos Ka - B \sin Ka \quad \text{—} ①$$

$$+LC e^{-La} = AK \sin Ka + B \cos Ka \quad \text{—} ②$$

$x = -a$.

and

$$C' e^{-La} = A \cos Ka + B \sin Ka \quad \text{—} ③$$

$$-LC' e^{-La} = -AK \sin Ka + BK \cos Ka$$

$x = a$.

$$\Rightarrow LC' e^{-La} = AK \sin Ka - BK \cos Ka \quad \text{—} ④$$

① + ③ $\Rightarrow \quad (C + C') e^{-La} = 2A \cos Ka$.

② + ④ $\Rightarrow \quad \frac{L}{K}(C + C') e^{-La} = 2A \sin Ka$

$$\Rightarrow \frac{L}{K} = \tan Ka .$$

$$\boxed{\Rightarrow \quad \cancel{2m} E_1 = (V_0 - E_1) \tan^2\left(\frac{1}{\hbar}\sqrt{2m(V_0 - E_1)}\right)}$$

$\Rightarrow$ This is an equ^n of E_1 and thus E_1 is fixed by the sol^n. Thus energy E_1 is quantized!

Thus we see that out of 4 eqn's, ① puts constraint on E_1.

Other 3 eqn's can be used to fix A, B, C' in terms of C.

(*) Thus the final wave fn will have one overall const C; This is perfectly ok, as Schrödinger eqn can not fix overall normalization. One can fix 'C' by demanding unit normalization.

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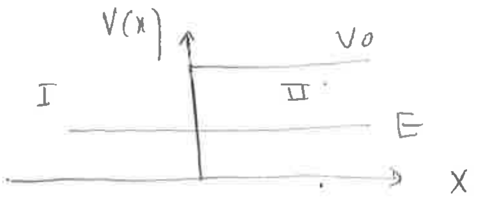
4)



Soln in I : $\psi(x) = A\cos kx + B\sin kx \qquad x < 0$

$$k = \frac{1}{\hbar}\sqrt{2mE}$$

Soln in II : $\psi(x) = De^{Lx} + Ce^{-Lx} \qquad x > 0,$

$$L = \frac{1}{\hbar}\sqrt{2m(V_0 - E)}$$

- Finiteness $\Rightarrow$ $D = 0$.

- Continuity of ψ & $\frac{\partial \psi}{\partial x}$ at $x = 0$ gives:

$$C = A, \qquad -LC = BK \Rightarrow B = -\frac{LC}{K}$$

Thus two condition fixes A & B interms of C. C is not fixed by Schrödinger eqn and can only be fixed by normalization condn.

- $\exists$ no constraint on E in that region, i.e in range $0 < E < V_0$, Φ all possible value of E is acceptable $\Longrightarrow$ no quantization!!