Assignment 2 [PHY202 (solutions)]

1. Mass of electron, proton, neutron and dust particle are as,

$$\begin{aligned} M_e &= 9.1 \times 10^{-31}\text{kg} \\ M_p &= 1.67 \times 10^{-27}\text{kg} \\ M_n &= 1.67 \times 10^{-27}\text{kg} \\ M_d &\sim 10^{-15}\text{kg} \end{aligned}$$

   (a) Velocity of the dust particle $v_d \sim 10^{-3}$m/sec. Its deBroglie wavelength

$$\lambda_{dust} = \frac{h}{p} = \frac{h}{M_d v_d} = \frac{6.626 \times 10^{-34}}{10^{-15} \times 10^{-3}}\text{m} \sim 10^{-15}\text{m}.$$

   (b) The neutron is at room temperture, we get

$$\frac{p^2}{2m} \sim k_B T \sim 1.4 \times 10^{-23} \times 300\text{Jule} \sim 10^{-20}\text{Jule}.$$

   Momentum of the thermal neutron $p \sim 10^{-23}$Jule sec/m. de Broglie wavelength $\lambda_n = \frac{h}{p} \sim 10^{-10}$m.

   (c) Notice that λ_{dust} is very small compared to the physical size of a dust particle ($\sim 10^{-6}$). But λ_n is $\sim 10^5$ larger than the size of a neutron ($\sim 10^{-15}$m).
   So, scattering of thermal neutron will have Quantum effect, compared to that of dust.

2.

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{\frac{ip_x.x}{\hbar}} \phi(p_x) dp_x$$

and

$$\int_{-\infty}^{\infty} |\phi(p_x)|^2 dp_x = 1$$

Now

$$\int_{-\infty}^{\infty}|\psi(x)|^2dx$$
$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dxdp_x^1dp_x^2\phi^*(p_x^1)\phi(p_x^2)e^{-i\frac{x}{\hbar}(p_x^1-p_x^2)}$$
$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dp_x^1dp_x^2\phi^*(p_x^1)\phi(p_x^2)\Big(2\pi\delta\Big(-\frac{(p_x^1-p_x^2)}{\hbar}\Big)\Big) \quad \text{performe the } x \text{ integral first}$$
$$= \int_{-\infty}^{\infty}dp_x^1dp_x^2\phi^*(p_x^1)\phi(p_x^2)\delta(p_x^1-p_x^2) \qquad \text{using } \delta(ax)=\frac{\delta(x)}{|a|}$$
$$= \int_{-\infty}^{\infty}dp_x^1\phi^*(p_x^1)\phi(p_x^1)$$
$$= 1$$

Now

$$\psi(x,t)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{\frac{i}{\hbar}(p_x.x-E(p_x)t)}\phi(p_x)dp_x$$

Do similar computation as earlier. We get

$$\int_{-\infty}^{\infty}|\psi(x,t)|^2dx$$
$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dxdp_x^1dp_x^2\phi^*(p_x^1)\phi(p_x^2)e^{-i\frac{x}{\hbar}(p_x^1-p_x^2)}e^{\frac{it}{\hbar}(E(p_x^1)-E(p_x^2))}$$
$$= \int_{-\infty}^{\infty}dp_x^1dp_x^2\phi^*(p_x^1)\phi(p_x^2)e^{\frac{it}{\hbar}(E(p_x^1)-E(p_x^2))}\delta(p_x^1-p_x^2)$$
$$= \int_{-\infty}^{\infty}dp_x^1\phi^*(p_x^1)\phi(p_x^1)$$
$$= 1$$

Finally,

$$\phi(p_x,t)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{\frac{-ip_x.x}{\hbar}}\psi(x,t)dx$$

Using similar trick, we find

$$\int_{-\infty}^{\infty}|\phi(p_x,t)|^2dp_x=1$$

3. Schrodinger equation

$$\Big[-\frac{\hbar^2}{2m}\vec{\nabla}^2+V(\vec{x},t)\Big]\psi(\vec{x},t)=i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t)$$

So,

$$\frac{\partial}{\partial t}\psi(\vec{x},t) = \frac{-i}{\hbar}\Big[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{x},t)\Big]\psi(\vec{x},t) \qquad (1)$$

$$\frac{\partial}{\partial t}\psi^*(\vec{x},t) = \frac{i}{\hbar}\Big[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{x},t)\Big]\psi^*(\vec{x},t) \qquad (2)$$

As $p(\vec{x},t)d^3x = |\psi(\vec{x},t)|^2 d^3x$,

$$\int_V p(\vec{x},t)d^3x = \int_V |\psi(\vec{x},t|^2 d^3x$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_V p(\vec{x},t)d^3x \\
&= \int_V \Big(\psi^*\frac{\partial\psi}{\partial t} + \frac{\partial\psi^*}{\partial t}\psi\Big)d^3x \\
&= \frac{i\hbar}{2m}\int_V \Big(\psi^*\nabla^2\psi - (\nabla^2\psi^*)\psi\Big)d^3x \\
&= \frac{i\hbar}{2m}\int_V \vec{\nabla}.\Big(\psi^*(\vec{\nabla}\psi) - (\vec{\nabla}\psi^*)\psi\Big)d^3x \\
&= \frac{i\hbar}{2m}\int_S \Big(\psi^*(\vec{\nabla}\psi) - (\vec{\nabla}\psi^*)\psi\Big).\vec{ds} \\
&= 0. \quad (\psi \text{ is localised in space time, it must vanish at the boundary.})
\end{aligned}$$

4. (a)

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\alpha u^2}e^{-\beta u}du &= e^{\frac{\beta^2}{4\alpha}}\int_{-\infty}^{\infty} e^{-\alpha\left(u+\frac{\beta}{2\alpha}\right)^2}du \\
&= e^{\frac{\beta^2}{4\alpha}}\int_{-\infty}^{\infty} e^{-\alpha x^2}dx \quad \text{using } x = u + \frac{\beta}{2\alpha}
\end{aligned}$$

Now,

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\alpha x^2}dx &= \sqrt{\int_{-\infty}^{\infty} e^{-\alpha x^2}dx\int_{-\infty}^{\infty} e^{-\alpha y^2}dy} \\
&= \sqrt{\int_0^{2\pi} d\theta\int_0^{\infty} dr r e^{-\alpha r^2}} \quad \text{moving to polar coordinate} \\
&= \sqrt{2\pi\Big(\frac{-1}{2\alpha}\Big)\int_0^{\infty}\frac{d}{dr}(e^{-\alpha r^2})dr} \\
&= \sqrt{\frac{\pi}{\alpha}}
\end{aligned}$$

So,

$$\int_{-\infty}^{\infty} e^{-\alpha u^2} e^{-\beta u} du = e^{\frac{\beta^2}{4\alpha}} \sqrt{\frac{\pi}{\alpha}}.$$

(b)

$$\begin{aligned}
&\int_{-\infty}^{\infty} x^n e^{-\alpha x^2} dx \quad \text{where } n = 2, 4, 6 \cdots \\
&= \int_{-\infty}^{\infty} x^{2m} e^{-\alpha x^2} dx \quad \text{considering } n = 2m \text{ where } m = 1, 2, 3 \cdots \\
&= (-1)^m \frac{d^m}{d\alpha^m} \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \\
&= (-1)^m \frac{d^m}{d\alpha^m} \left(\sqrt{\frac{\pi}{\alpha}} \right) \\
&= (-1)^{2m} \pi^{1/2} \frac{1.3.5 \cdots (2m-1)}{2^m \alpha^{\frac{2m+1}{2}}} \\
&= \frac{1.3.5 \cdots (n-1)}{2^{n/2} \alpha^{\frac{n+1}{2}}} \pi^{1/2}
\end{aligned}$$

5. (a)

$$\begin{aligned}
\delta(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} dp \\
\delta_\epsilon(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} e^{-\epsilon p^2} dp, \quad \epsilon > 0 \\
&= \frac{1}{2\sqrt{\pi\epsilon}} e^{-\frac{x^2}{4\epsilon}} \quad \text{using previous problem's result}
\end{aligned}$$

It is a Gaussian function of width $2\sqrt{\epsilon}$. In the limit $\epsilon \to 0$, this function vanishes everywhere except $x = 0$, where it is diverging. Thus in the limit $\epsilon \to 0$, $\delta_\epsilon(x)$ is a delta function.

(b)

$$\int_{-\infty}^{\infty} \delta(ax) dx = \int_{-\infty}^{\infty} \delta(y) \frac{dy}{|a|} = \frac{1}{|a|}$$

So, $\delta(ax) = \frac{\delta(x)}{|a|}$

6. Heisenberg uncertainity principle,

$$\Delta p = \frac{\hbar}{2\Delta x} \sim 10^{-17} \text{Jule sec/m} \tag{3}$$

Compared to thermal neutron, it is 10^6 times larger.

7. Minimum momentum

$$\Delta p = \frac{\hbar}{2\Delta x} \sim \frac{10^{-34}}{2 \times 5 \times 10^{-5}} \text{Jule sec/m} \sim 10^{-28} \text{Jule sec/m}. \qquad (4)$$

So, energy$\sim \frac{(\Delta p)^2}{2m} \sim 10^{-52}$ Jule.