Topology Comprehensive
June, 2015

1. (a) Let $E = \mathbb{R}^2 \setminus \{(0,0)\}$ and let η be the 1-form

$$\eta = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}.$$

   i. Prove that η is closed in E.

   ii. Let $r > 0$ and consider the curve $\gamma : [0, 2\pi]$ defined by

$$\gamma(t) = (r\cos t, r\sin t).$$

   Prove that $\int_\gamma \eta = 2\pi$, and also show that η is not exact in E.

   (b) Let I be the unit interval $[0,1] \subset \mathbb{R}$ and suppose $\gamma : I \to I \times I$ be a continuous, surjective map.

   i. Can γ be injective?

   ii. Can γ be C^∞? Recall that a C^∞ map between I and $I \times I$ is a restriction of a C^∞ map on some open set containing I.

2. (a) Let D^2 be the standard unit disc in the plane, and let

$$A = \{(1,0), (0,1), (-1,0)\}$$

   be the set consisting of the designated points on the boundary circle S^1. Let X be the quotient space D^2/A, and let B be the subspace S^1/A.

   i. Describe a CW structure on X in which B is a subcomplex. Your description should include a list of cells used, a clearly labelled diagram and a written description of the attaching maps.

   ii. Calculate the fundamental group of X.

   iii. Show that B is not a retract of X.

   (b) i. Give an example of a pair of path connected topological spaces $X \subset Y$ such that $\pi_1(X)$ is trivial but $\pi_1(Y)$ is not trivial. Justify your example.

   ii. Give an exaple of a pair of path connected topological spaces $X \subset Y$ such that $\pi_1(Y)$ is trivial but $\pi_1(X)$ is not trivial. Justify your example.

3. (a) Compute the homology of a Möbius strip. Using the fact that $\mathbb{RP}^2 = M \cup \mathbb{D}^2$, where the boundary circle of the Möbius strip is identified with the boundary circle of the disk via the identity map. Compute the homology groups of $\mathbb{RP}^2$.

(b) Compute the cohomology groups $H^*(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z})$ with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

(c) Compute the cohomology groups $H^*(\mathbb{RP}^2 \times \mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z})$.

(d) Consider the map $f : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ given by

$$f(x_1, x_2, \ldots, x_{2k-1}, x_{2k}) = (x_2, -x_1, \ldots, x_{2k}, -x_{2k-1}).$$

Let $\bar{f}$ be the induced map on $\mathbb{RP}^{2k-1} \to \mathbb{RP}^{2k-1}$. What is $\tau(\bar{f})$?

4. Suppose M is a closed, connected, oriented, 3 manifold with $\pi_1(M) \cong F_2$, the free group generated by 2 symbols. Compute $H_*(M, \mathbb{Z})$ and $H^*(M, \mathbb{Z})$.

5. Suppose $p : X \to Y$ and $q : Y \to Z$ be covering maps. Suppose q is such that inverse image of any point in Z is a finite set. Prove that $q \circ p$ is a covering map. Write down a relationship between the deck transformation groups of p, q and $q \circ p$.

6. (a) For k, l positive integers, compute the integral cohomology ring of $S^k \times S^l$.

(b) For $n > 0 \in \mathbb{Z}$, compute the integral cohomology ring of S^n.

(c) Prove that any continuous map $f : S^{k+l} \to S^k \times S^l$ induces a zero homomorphism $H^{k+l}(S^k \times S^l, \mathbb{Z}) \to H^{k+l}(S^{k+l}, \mathbb{Z})$.