Write your answers in the answer sheets provided. Give full explanation with clear statements of any theorem you use. Use no books or notes in this exam. Attempt all problems. You have 3 hours.

1. Let $\left\{ {n \atop k} \right\}$ denote the number of equivalence relations on a set of cardinality n, having exactly k equivalence classes. Prove the following identities:

   (a) (10 points)
   $$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} x(x-1)\dots(x-k+1) = x^n.$$

   (b) (10 points)
   $$\left\{ {n \atop k} \right\} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n.$$

   (Hint: derive a recurrence relation for $\left\{ {n \atop k} \right\}$ in terms of $\left\{ {n-1 \atop i} \right\}$'s and use induction.)

2. (10 points) Derive a closed form formula for the number of derangements of a set of cardinality n.

3. (15 points) If G is a bridge less 3-regular graph then prove that it has a perfect matching. Prove that every bridge less 3-regular graph G can be expressed as the edge sum of a 1-factor and a 2-factor.

4. (a) (8 points) Prove that a connected k-regular bipartite graph is 2-connected.

   (b) (7 points) Let $\alpha(G)$ denote the size of a maximum independent set in G. Then prove that a graph G with all degrees at most d satisfies
   $$\alpha(G) \geq \frac{|V|}{d+1}$$

5. (a) (10 points) Let $A[i] : i = 0, 1, 2, \dots, n-1$ be an array of n distinct integers. We wish to sort A in decreasing order. We are given that each element in the array is at a position that is at most k away from its position in the sorted array, that is, we are given that $A[i]$ will move to a position in $\{i-k, i-k+1, \dots, i, \dots, i+k-1, i+k\}$ after the array is sorted in decreasing order. Here is one proposal for sorting A:

   Use heap sort, but with the following modification. Insert the first k elements of the array into a heap; delete max; insert the next element from the array into the heap; delete max; and so on.

   State precisely why this method always sorts the elements of the array correctly? How many comparisons will this method make in the worst case for such inputs? (accounting for the delete max and inserts, using big-Oh).

   (b) (5 points) Consider the following method of merging two sorted arrays a and b into c: Perform a binary search for $b[0]$ in the array a. If $b[0]$ is between $a[i]$ and $a[i+1]$, output $a[0]$ through $a[i]$ to the array c, then output $b[0]$ to the array c. Next perform a binary search for $b[1]$ in the subarray $a[i+1]$ to $a[n-1]$, where n is the number of elements in the array a, and repeat the output process. Repeat this procedure for every element of the array b. In which cases is this method more efficient than the method taught in the class? In which cases is it less efficient?

(c) (10 points) Show how to implement a first-in first-out queue with priority queue. Show how to implement a stack with a priority queue.

6. (a) (8 points) The edge connectivity of an undirected graph is the minimum number k of edges that must be removed to disconnect the graph. Show how the edge conncetivity of an undirected graph $G = (V, E)$ can be determined by a max-flow algorithm on atmost $|V|$ flow networks, each having $O(V)$ vertices and $O(E)$ edges.

   (b) (7 points) Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

   Let G be an arbitrary flow network, with a sours s and a sink t, and a positive integer capacity c_e on every edge e; and let (A, B) be a minimum $s - t$ cut with respect to these capacities $\{c_e : e \in E\}$. Now suppose we add 1 to every capacity; then (A, B) is still a minimum $s - t$ cut with respect to these new capacities $\{1 + c_e : e \in E\}$.