IISER PUNE MATHEMATICS COMPREHENSIVE EXAM, FALL 2015
ANALYSIS

All six questions are compulsory and carry equal weight. Maximum possible score is 60.

(Q.1) Suppose B is a bounded linear operator on a Hilbert space H. Show that there exists a real number α such that $B - \alpha$ is invertible i.e. it is bijective and its inverse is also a bounded linear operator on H.

(Q.2) Let $\epsilon > 0$ and suppose $f : \mathbb{R} \to (0, \infty)$ is a continuous function which satisfies the following properties:

(a) $\int_{\mathbb{R}} f(x)\, dx = 1$.
(b) $\int_{\mathbb{R}} x^2 f(x)\, dx = \epsilon$.

Let μ be the Borel measure defined on $\mathbb{R}$ by $\mu(E) = \int_E f(x)\, dx$ for any Borel subset $E \subseteq \mathbb{R}$. Show that $\mu(\delta, \infty) \leq \epsilon/\delta^2$ for any $\delta > 0$.

(Q.3) Let $\mathbb{C}[t]$ be the space of all polynomials in one variable with coefficients in $\mathbb{C}$. Define a norm $\|\ \|$ on $\mathbb{C}[t]$ by,
$\|0\| := 0$; and $\|P(t)\| :=$ max $\{|a_j| : 0 \leq j \leq n.\}$ where $P(t) = a_0 + a_1 t + \cdots + a_n t^n$ such that $a_n \neq 0$.
Prove or disprove: $(\mathbb{C}[t], \|\ \|)$ is a Banach space.

(Q.4) Show that the sequence $\{\sin\ nx\}_{n=1}^{\infty}$ has no limit point in the Hilbert space $L^2\,([-\pi, \pi], dm)$ where dm is the usual Lebesgue measure on the Borel σ-algebra of $[-\pi, \pi]$.

(Q.5) Let $f : \mathbb{C} \to \mathbb{C}$ be a function which is holomorphic on the open ball $B(0,2)$ and satisfies the following condition:

$$|f(z)| < 1 \quad \text{for all } z \text{ such that } |z| = 1.$$

Prove that f has a unique fixed point in the open ball $B(0,1)$.

(Q.6) Consider the following integral:

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt \quad \forall\ s \in \mathbb{C} \text{ such that } Re(s) > 0$$

This defines a holomorphic function in the half plane $Re(s) > 0$ which can be extended to a meromorphic function on $\mathbb{C}$ using analytic continuation. Show that for any positive integer n, the function Γ has a pole at $s = -n$ and find its residue at $s = -n$.