Algebra Comprehensive Examination

2015

May 29, 2015

You can get maximum 100 marks and 180 minutes!

1. Let p be a prime integer and let G be a finite p-group. Let $Z(G)$ be the center of the group G and $C \trianglelefteq Z(G)$ be the subgroup consisting of all elements x satisfying $x^p = 1$. Let N be a normal subgroup of G such that $N \cap C = \{1\}$. Prove that $N = \{1\}$.

2. Let A be a ring and $\mathfrak{P}_1, \dots, \mathfrak{P}_r$ be prime ideals in A. Let $I = \langle a_1, \dots, a_n \rangle$ be an ideal of A such that $I \nsubseteq \mathfrak{P}_i$, $1 \le i \le r$. Prove that there exist $b_2, \dots, b_n \in A$ such that the element

$$c = a_1 + a_2 b_2 + \cdots + a_n b_n \notin \cup_{i=1}^{r} \mathfrak{P}_i.$$

[Hints:-Prove by induction on r.]

3. Prove or disprove the following statements:

(i) The tensor product of two field is always a field.

(ii) Let R be an integral domain. For any torsion R-module M, the annihilator of M is always non-zero.

4. Let $0 \to K \to P \to M \to 0$ and $0 \to K' \to P' \to M \to 0$ are short exact sequences of R-modules with P and P' are projective, then prove that $K \oplus P'$ is isomorphic to $K' \oplus P$.

[Hints:- Define the following submodule of $P \oplus P'$, where $\Phi : P \to M$ and $\Phi' : P' \to M$:

$$X = \{(p, q) \in P \oplus P' : \Phi(p) = \Phi'(q)\}.]$$

5. Let L/K be an algebraic field extension. Let $\lambda \in L$ be nonzero and such that λ and λ^2 have the same minimal polynomial over K. Prove that λ is a root of unity.

6. Consider the polynomial $f(x) = x^3 - x^2 - 2x + 1$. Let K be the splitting field of f. Find the Galois group of K over $\mathbb{Q}$.

[Hints:-The discriminant of $g(y) = y^3 + py + q$ is $D = -4p^3 - 27q^2$.]

Best of luck!