Topology Comprehensive
June, 2016

In all the homology and cohomology computations below, the base ring is $\mathbb{Z}$, unless specified.

Question 1 (6 points)

(a) Which of the following 1-forms in $\mathbb{R}^2$ is exact? If they are exact write them as the exterior derivative of some function.

i. $e^y dx + x(1 + e^y)dy$,

ii. $(x^2 + y^2)dx + 2xydy$.

(b) Let S be a surface in $\mathbb{R}^3$ such that S intersects the upper half space in a half sphere

$$S \cap \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, z \geq 0\right\} = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\right\},$$

as in Figure 1. Let $T = S \cap \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, z \leq 0\right\}$, and let $\iota : T \to \mathbb{R}^3$ be the inclusion. Consider the 2-form in $\mathbb{R}^3$

$$\omega = ydx \wedge dz + zdx \wedge dy.$$

Compute

$$\int_T \iota^* \omega.$$

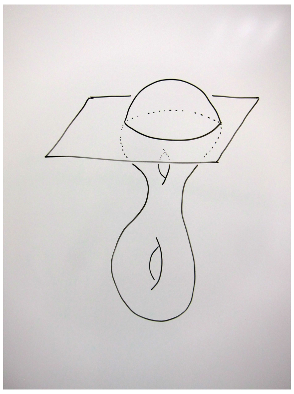

Figure 1: S (problem 1(b))

Question 2 (6 points)

(a) A dunce cap is the topological space obtained by identifying the sides of a triangle as shown in Figure 2. Find the fundamental group of the dunce cap.

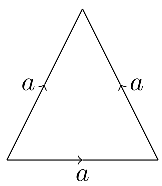

Figure 2: Dunce cap (problem 2(a))

(b) Calculate $\pi_n(S^1)$ for $n \geq 2$.

QUESTION 3 (6 points)

(a) Define the chain complex and boundary maps of cellular homology.

(b) Calculate all the homology groups of the space X obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.

QUESTION 4 (6 points)

(a) State the Mayer-Vietoris theorem.

(b) Calculate $H_1(S^3 \setminus K, \mathbb{Z})$ where $K \subset S^3$ is the trefoil knot shown below (Figure 3). (Hint: Use a regular neighbourhood of the knot in $\mathbb{R}^3$.)

Figure 3: Trefoil knot (problem 4(b))

QUESTION 5 (6 points)

(a) Let M be a closed, connected, oriented 3-manifold. Let $H_1(M, \mathbb{Z}) = \mathbb{Z}^r \oplus F$ where r is the rank of $H_1(M, \mathbb{Z})$ and F is a finite abelian group. Express all other homology and cohomology groups of M in terms of r and F.

(b) With M being as above, compute $H^2(M, \frac{\mathbb{Z}}{2\mathbb{Z}})$.

QUESTION 6 (6 points)

(a) Let F_g be a closed, oriented surface of genus $g > 1$. Show that any homeomorphism of F_g that is homotopic to the identity must have a fixed point.

(b) Calculate the cohomology ring of $\mathbb{R}P^2 \times S^1$.