Average Orders of Certain Arithmetical Functions

Kaneenika Sinha*

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Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada K7L 3N6,
email: skaneen@mast.queensu.ca

ABSTRACT. We consider the functions $M(n)$, the maximum exponent of any prime power dividing n and $m(n)$, the minimum exponent of any prime power dividing n. The sums $\sum_{n\leq x} M(n)$ and $\sum_{n\leq x} m(n)$ have been well investigated in the literature. In this note, we will improve known estimates of both the above sums under the assumption of the Riemann hypothesis. We will also obtain Ω-type estimates for these sums unconditionally.

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1 Introduction

A natural number $n \geq 2$ can be written uniquely as a product of prime numbers,

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k},\ \alpha_i \geq 1.$$

We define

$$M(n) = \max\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$$

and

$$m(n) = \min\{\alpha_1, \alpha_2, \cdots, \alpha_k\}.$$

We also define $M(1) = m(1) = 1$. The sums $\sum_{n\leq x} M(n)$ and $\sum_{n\leq x} m(n)$ have been well investigated in the literature. In 1969, Ivan Niven [10] proved that

$$\sum_{n\leq x} M(n) \sim Bx, \text{ as } x \to \infty \tag{1}$$

where

$$B = 1 + \sum_{k=2}^{\infty} \left(1 - \frac{1}{\zeta(k)}\right).$$

B can be evaluated to be 1.705211 approximately. He also showed that

$$\sum_{n\leq x} m(n) = x + \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)} x^{\frac{1}{2}} + \mathrm{o}(x^{\frac{1}{2}}). \tag{2}$$

The aim of this note is to prove the following theorems:

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Theorem 1 *Assuming the Riemann hypothesis,*

$$\sum_{n\le x} M(n) = Bx + \mathrm{O}(x^{\frac{17}{54}+\epsilon}). \tag{3}$$

Moreover, unconditionally, we have

$$\sum_{n\le x} M(n) = Bx + \Omega(x^{\frac{1}{4}}). \tag{4}$$

Theorem 2 *Assuming the Riemann hypothesis, we have*

$$\sum_{n\le x} m(n) = x + \sum_{i=2}^{7} A(i)x^{\frac{1}{i}} + \mathrm{O}(x^{\frac{12}{85}+\epsilon}), \tag{5}$$

where

$$\gamma_{i,k} = \prod_{\substack{0\le j\le k-1\\ j\ne i}} \zeta\left(\frac{k+j}{k+i}\right) \prod_p \left(1 + \sum_{m=k+1}^{2k-1} p^{-\frac{m}{k+i}} - \sum_{m=2k+2}^{3k} p^{-\frac{m}{k+i}}\right),$$

$$A(2) = \gamma_{0,2},\ A(3) = \gamma_{1,2} + \gamma_{0,3},\ A(4) = \gamma_{1,3} + \gamma_{0,4},$$

$$A(5) = \gamma_{2,3} + \gamma_{1,4} + \gamma_{0,5},\ A(6) = \gamma_{2,4} + \gamma_{1,5} + \gamma_{0,6},$$

and

$$A(7) = \gamma_{3,4} + \gamma_{2,5} + \gamma_{1,6} + \gamma_{0,7}.$$

Also, unconditionally, we have

$$\sum_{n\le x} m(n) = x + \sum_{i=2}^{7} A(i)x^{1/i} + \Omega(x^{\frac{1}{10}}). \tag{6}$$

Before proving these theorems, we would like to briefly review what is known about these sums. Perhaps unaware of Niven's work, in 1975, C. W. Anderson proposed the following problem in the American Mathematical Monthly:

Problem: *Prove that*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} M(n) = 1 + \sum_{m=2}^{\infty} \frac{\mu(m)}{m(m-1)},$$

where $\mu(m)$ is the Möbius function.

In response, T. Salat pointed out that the above sum had already been investigated by Niven in his 1969 paper. A solution to Anderson's problem was also submitted by Ram Murty and Kumar Murty. In fact, they reported a mistake in the question and showed that the correct term on the right hand side should be

$$1 - \sum_{m=2}^{\infty} \frac{\mu(m)}{m(m-1)},$$

but this was not mentioned by the editor when responses to Anderson's problem were referenced in a later issue of the Monthly (see [1].) It is not difficult to see that this correct term is indeed equal to Niven's constant B. Since

$$\left(1 - \frac{1}{\zeta(k)}\right) < \frac{1}{2^{k-1}} \text{ for } k \ge 2,$$

the sum defining B in (1) is absolutely convergent. Thus, by interchanging summations, we see that

$$\sum_{k=2}^{\infty}\left(1-\frac{1}{\zeta(k)}\right)=-\sum_{k=2}^{\infty}\sum_{m=2}^{\infty}\frac{\mu(m)}{m^k}=-\sum_{m=2}^{\infty}\frac{\mu(m)}{m(m-1)}.$$

Niven's result was made effective by D. Suryanarayana and R. Sitaramachandrarao in 1977. In [12], they proved that

$$\sum_{n\le x} M(n)=Bx+x^{\frac{1}{2}}\exp(-c\log^{3/5}x(\log\log x)^{-\frac{1}{5}}), \tag{7}$$

where c is an absolute positive constant. The main ingredient in their proof is a result of Walfisz (proved in [14]), which we will state in the next section. They also prove that

$$\begin{aligned}\sum_{n\le x} m(n)&=x+\gamma_{0,2}x^{\frac{1}{2}}+(\gamma_{1,2}+\gamma_{0,3})x^{\frac{1}{3}}+(\gamma_{1,3}+\gamma_{0,4})x^{\frac{1}{4}}\\&\quad+(\gamma_{2,3}+\gamma_{1,4}+\gamma_{0,5})x^{\frac{1}{5}}+\mathrm{O}(x^{\frac{1}{6}}),\end{aligned} \tag{8}$$

where $\gamma_{i,k}$'s are as defined in Theorem 2.

Unaware of the results of [12], in 1991, Hui Zhong Cao obtained an error estimate of $\mathrm{O}(x^{\frac{1}{2}}\log x)$ in [4]. This estimate is weaker than that obtained in [12]. Cao also provides an estimate for $\sum_{n\le x} m(n)$, but this is also not as sharp as equation (8). In the next two sections, we will estimate $\sum_{n\le x} M(n)$. We will also refine some known estimates about $\sum_{n\le x} m(n)$ in the last section. The two main theorems proved in this note are Theorems 1 and 2.

2 Preliminary results

In this section, we will mention some results which are needed in this paper. For a fixed positive integer $k\ge 2$, we define a k-free integer as a positive integer which is not divisible by the k-th power of any of its prime factors. Let $S_k(x)$ denote the number of k-free integers less than or equal to x. It turns out that $\sum_{n\le x} M(n)$ is directly related to the distribution of k-free numbers. By elementary number theory we know that for an integer $k\ge 2$,

$$S_k(x)=\frac{x}{\zeta(k)}+\mathrm{O}(x^{\frac{1}{k}}).$$

The direct connection between the error terms

$$E_k(x)=S_k(x)-\frac{x}{\zeta(k)}$$

for $k\ge 2$ and the error term in the sum $\sum_{n\le x} M(n)$ is indicated in the following proposition, which is also mentioned in [10] and [12]:

Proposition 3

$$\sum_{n\le x} M(n)=Bx-\sum_{k=2}^{j}E_k(x)+\mathrm{O}(1),$$

where

$$j=\left\lceil\frac{\log x}{\log 2}\right\rceil.$$

Proof. We begin by observing that the maximum of $\{M(1), M(2), \cdots M(n)\}$ is $\left\lceil \frac{\log x}{\log 2} \right\rceil$. Also, for $k \geq 2$, the number of integers less than or equal to x such that $M(n) = k-1$ is $S_k(x) - S_{k-1}(x)$. Thus, if

$$j = \left\lceil \frac{\log x}{\log 2} \right\rceil,$$

we get that

$$\sum_{n \leq x} M(n) = \sum_{k=2}^{j+1} (k-1)(S_k(x) - S_{k-1}(x)).$$

Since $S_{j+1}(x) = x$, the above sum is equal to

$$jx + \sum_{k=2}^{j} (k-1) S_k(x) - \sum_{k=1}^{j} k S_k(x),$$

which, on further simplification, is equal to

$$jx - \sum_{k=2}^{j} S_k(x) - 1.$$

Writing

$$S_k(x) = \frac{x}{\zeta(k)} + \mathrm{O}(x^{\frac{1}{k}}),$$

the above expression becomes

$$x + x \sum_{k=2}^{j} \left(1 - \frac{1}{\zeta(k)}\right) - \sum_{k=2}^{j} E_k(x) - 1.$$

This is equal to

$$Bx - x \sum_{k>j} \left(1 - \frac{1}{\zeta(k)}\right) - \sum_{k=2}^{j} E_k(x) - 1.$$

Since

$$1 - \frac{1}{\zeta(k)} < \frac{1}{2^{k-1}},$$

we get that

$$\sum_{k>j} \left(1 - \frac{1}{\zeta(k)}\right) < \sum_{k>j} \frac{1}{2^{k-1}} = \frac{1}{2^{j-1}} = \mathrm{O}\left(\frac{1}{x}\right).$$

Thus,

$$\sum_{n \leq x} M(n) = Bx - \sum_{k=2}^{j} E_k(x) + \mathrm{O}(1).$$

□

While the trivial estimate for $E_k(x)$ is $\mathrm{O}(x^{\frac{1}{k}})$, substantially better estimates can be found, especially under the assumption of the Riemann hypothesis. We refer the reader to an excellent survey of Pappalardi [11] on various improvements of this error term. In particular, the following results are relevant to this note:

In 1963, Walfisz [14] proved that

$$E_k(x) = x^{\frac{1}{k}} \exp(-ck^{-8/5} \log^{3/5} x (\log\log x)^{-1/5}). \tag{9}$$

We advise the reader that Walfisz's theorem has not been recorded correctly in [11]. The exponent 1/5 in the third displayed formula on page 73 should be replaced by $-1/5$. The authors, in [12], prove (7) by applying Walfisz's result (9) for $k = 2$ and the trivial estimate $\mathrm{O}(x^{\frac{1}{k}})$ for $k \geq 3$ to the right hand side of Proposition 3. Clearly, any improvement in the estimates for $E_k(x)$ will give a better error term for $\sum_{n\leq x} M(n)$. We now state two better estimates for $E_k(x)$ which have been obtained by assuming the Riemann hypothesis. In 1993, Jia [8] proved the following result:

Proposition 4 *Assuming the Riemann hypothesis,*

$$E_2(x) = \mathrm{O}(x^{17/54+\epsilon}).$$

The next result was proved in 1981 by Montgomery and Vaughan [9]:

Proposition 5 *Assuming the Riemann hypothesis, for $k \geq 2$,*

$$E_k(x) = \mathrm{O}(x^{\frac{1}{k+1}+\epsilon}) \textit{ for any } \epsilon > 0.$$

In 1989, Graham and Pintz [5] obtained the following better estimate for $E_3(x)$:

Proposition 6 *Assuming the Riemann hypothesis,*

$$E_3(x) = \mathrm{O}(x^{7/30}).$$

3 Proof of Theorem 1

We recall, from Proposition 3, we get that

$$\sum_{n\leq x} M(n) = Bx - \sum_{k=2}^{j} E_k(x) + \mathrm{O}(1).$$

By Proposition 4,

$$E_2(x) = \mathrm{O}(x^{17/54+\epsilon}).$$

Moreover, by Proposition 5, for $3 \leq k \leq j$,

$$E_k(x) = \mathrm{O}(x^{\frac{1}{4}+\epsilon}).$$

Thus, we get

$$\sum_{n\leq x} M(n) = Bx + \mathrm{O}(x^{17/54+\epsilon}).$$

This proves equation (3), the first part of Theorem 1. In order to obtain equation (4), it suffices to show that

$$E_2(x) + E_3(x) = \Omega(x^{\frac{1}{4}}). \tag{10}$$

This is because equation (9) gives us an unconditional estimate for $E_4(x)$, which is sharper than $\mathrm{O}(x^{1/4})$. Substituting the trivial estimate $\mathrm{O}(x^{1/k})$ for $k \geq 5$, equation (9) for $k = 4$ and equation (10) in Proposition 3, we obtain equation (4). Equation (10) follows as a corollary to the following general result, which seems not to have been noticed before and is interesting in its own right:

Proposition 7 *Let $S(x)$ be a complex-valued function. Suppose that*

$$f(s) = s\int_1^\infty \frac{S(x)}{x^{s+1}}dx$$

has a pole at a complex number with real part θ. Then

$$S(x) = \Omega(x^\theta).$$

Proof. The main idea of the proof of Proposition 7 is inspired by a technique followed by Vaidya in [13] to find out an Ω-type estimate for $E_k(x)$ conditional upon either the Riemann hypothesis or the existence of a simple zero of $\zeta(ks)$ on the line $\mathrm{Re}(s) = k/2$ which is not a zero of $\zeta(s)$.

Let us assume that $S(x) = \mathrm{o}(x^\theta)$. Then, for every $\epsilon > 0$, we can find $x_0 = x_0(\epsilon)$ such that for $x \geq x_0$, we have $|S(x)| < \epsilon x^\theta$. Let $|S(x)| \leq A$ for $0 < x < x_0$. Thus, for $s = \sigma + it$ with $\sigma > \theta$, we have

$$\begin{aligned} |f(s)| &< |s|A\int_1^\infty x^{-1-\theta}dx + |s|\epsilon\int_1^\infty x^{-\sigma-1+\theta}dx \\ &= |s|\frac{A}{\theta} + |s|\frac{\epsilon}{\sigma-\theta}. \end{aligned}$$

Now, let us first consider the case when $f(s)$ has a simple pole at $\rho = \theta + it_0$ with non-zero residue c. Then, as $\sigma \to \theta, \quad (t = t_0)$, we get that

$$\lim_{\sigma\to\theta} |(\sigma-\theta)f(\sigma+it_0)| = |c|.$$

Thus, by the above inequality, we get that for every $\epsilon > 0$,

$$\begin{aligned} |c| &= \lim_{\sigma\to\theta} |(\sigma-\theta)f(\sigma+it_0)| \\ &\leq \lim_{\sigma\to\theta} \left\{ |\sigma+it_0|(\sigma-\theta)\frac{A}{\theta} + |\sigma+it_0|\epsilon \right\} \\ &= \epsilon|\sigma+it_0|. \end{aligned}$$

This is not possible as c is non-zero. Thus, our assumption that $S(x) = \mathrm{o}(x^\theta)$ is false. Therefore,

$$S(x) = \Omega(x^\theta).$$

The case when $f(s)$ has a pole of order ≥ 2 can also be dealt with by a similar analysis. This proves Proposition 7. □

In particular, if we let

$$S(x) = E_2(x) + E_3(x),$$

then it is not difficult to see that

$$s\int_1^\infty \frac{S(x)}{x^{s+1}}dx = \frac{\zeta(s)}{\zeta(2s)} + \frac{\zeta(s)}{\zeta(3s)} - \frac{\zeta(s)}{\zeta(2)} - \frac{\zeta(s)}{\zeta(3)}. \tag{11}$$

Let $\varrho = \frac{1}{4} + i\gamma$ be a pole of the right hand side of equation (11). We prove the existence of such a pole by applying some zero-density arguments. A classical result of Selberg tells us that for T

sufficiently large, the number of zeros of $\zeta(2s)$ on the line segment joining $\frac{1}{4}$ to $\frac{1}{4}+iT$ is at least $AT\log T$ for some $A>0$. Such zeroes are the singularities of

$$\frac{\zeta(s)}{\zeta(2s)}+\frac{\zeta(s)}{\zeta(3s)}-\frac{\zeta(s)}{\zeta(2)}-\frac{\zeta(s)}{\zeta(3)}$$

unless possibly they are also zeros of $\zeta(s)$ or $\zeta(3s)$. Since the real part of ϱ is $1/4$, by the functional equation, we get a zero of $\zeta(s)$ where the real part of s is equal to $3/4$. However, by a theorem of Carlson, (see ([6], Theorem 10.1) we know that the number of zeros of $\zeta(s)$ up to height T with real part $3/4$ is $\mathrm{O}(T^{3/4}\log T)$. Therefore, not every zero of $\zeta(2s)$ is a zero of $\zeta(s)$ or $\zeta(3s)$. Thus, the function

$$s\int_1^{\infty}\frac{E_2(x)+E_3(x)}{x^{s+1}}dx$$

will have at least one pole with real part $1/4$. Thus, by applying Proposition 7 to the function $S(x)=E_2(x)+E_3(x)$, we deduce equation (10). This concludes the proof of Theorem 1.

4 Proof of Theorem 2

In this section, we refine known estimates of the sum $\sum_{n\le x}m(n)$. We recall that equation (8) has been proved unconditionally in [12]. Once again, this estimate can be improved by assuming the Riemann hypothesis. Analogous to a k-free integer, a k-full integer is defined as a positive integer which is divisible by the k-th power of all its prime factors. We observe that (analogous to Proposition 3), if $L_k(x)$ denotes the number of k-full integers less than or equal to x, then

$$\sum_{n\le x}m(n)=x+L_2(x)+\cdots L_7(x)+\mathrm{O}(x^{\frac{1}{8}}), \tag{12}$$

since $j=\mathrm{O}(\log x)$ and $\sum_{k\ge 8}L_k(x)=\mathrm{O}(x^{1/8})+\mathrm{O}(x^{1/9}\log x)$. We now state some known results about the distribution of k-full integers. The following result has been proved in [7]:

Proposition 8 *For every $k\ge 1$,*

$$L_k(x)=\gamma_{0,k}x^{1/k}+\cdots+\gamma_{k-1,k}x^{1/(2k-1)}+\Delta_k(x),$$

where the $\gamma_{i,k}$'s are as given after equation (8). Moreover,

$$\Delta_3(x)=\mathrm{O}(x^{\frac{263}{2052}}),\ \Delta_4(x)=\mathrm{O}(x^{\frac{3091}{25981}}),\ \Delta_5(x)=\mathrm{O}(x^{\frac{1}{10}})\ \textit{and}\ \Delta_6(x)=\mathrm{O}(x^{\frac{1}{12}}).$$

The following conditional result proved in [15] in 2001 gives an estimate for $\Delta_2(x)$:

Proposition 9 *Assuming the Riemann hypothesis,*

$$\Delta_2(x)=\mathrm{O}(x^{\frac{12}{85}+\epsilon})\ \textit{for any}\ \epsilon>0.$$

We also recall the following unconditional Ω-type result of Balasubramanian, Ramachandra and Subbarao proved in [3]:

Proposition 10

$$\Delta_2(x)=\Omega(x^{1/10}).$$

We are now ready to prove Theorem 2:

We had observed above in equation (12) that

$$\sum_{n\leq x} m(n) = x + L_2(x) + \cdots + L_7(x) + \mathrm{O}(x^{\frac{1}{8}}).$$

Thus, by using Proposition 9 for $k=2$, Proposition 8 for $3 \leq k \leq 6$, and the elementary estimate

$$\Delta_k(x) = \mathrm{O}(x^{\frac{1}{k+1}})$$

for $k=7$, we get that

$$\sum_{n\leq x} m(n) = x + \sum_{i=2}^{7} A(i) x^{\frac{1}{i}} + \mathrm{O}(x^{\frac{12}{85}+\epsilon}),$$

where the $A(i)$'s are as given above. We also observe that by applying equation (8), Proposition 8 and Proposition 10 to the right hand side of equation (12), we get that

$$\sum_{n\leq x} m(n) = x + \sum_{i=2}^{7} A(i) x^{1/i} + \Omega(x^{\frac{1}{10}}).$$

This proves Theorem 2.

5 Concluding remarks

Thus, in this paper, we saw how closely the Riemann hypothesis is linked to the averages of exponents in factoring integers. The key ingredients in the proofs of Theorems 1 and 2, as we saw in the last two sections, are the refined estimates of $E_k(x)$ for $k \geq 2$ obtained by assuming this ubiquitous conjecture of Riemann. We should note that these estimates were not available to Suryanarayana and Sitaramachandrarao when they wrote their paper [12] in 1977. One would also like to know if the estimates for $\Delta_k(x)$ for $k \geq 3$ can be further improved under the assumption of the Riemann hypothesis or even some quasi-Riemann hypothesis. Could one also obtain optimal error terms for the sums $\sum_{n\leq x} M(n)$ and $\sum_{n\leq x} m(n)$? We relegate these questions to future research.

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