Sept 14 - 2015

Recall: A function $T: V \to W$ between V/s V, W is said to be a lin. transf if $T(av_1 + bv_2) = aT(v_1) + bT(v_2)$ for all $a, b \in \mathbb{R}$, $v_1, v_2 \in V$.

We say V is finite dim'l if $\exists$ a lin transf.

- $\varphi: \mathbb{R}^m \to V$ which is bijective (bijective lin trans also called ISOMORPHISM)

In this case we define $\{\varphi(e_1), \ldots, \varphi(e_m)\}$ to be a basis of V, and $m := \dim(V)$.

Note: if $\psi: \mathbb{R}^n \to V$ is another isomorphism, then

$\psi^{-1} \circ \varphi: \mathbb{R}^m \to \mathbb{R}^n$ is isom. (bijective lin transf)

hence $m = n$.

Thus $\dim(V)$ is well defined.

Def: If no such $\varphi: \mathbb{R}^m \to V$ (isomorphism) exists for any $m \in \mathbb{N}$, we say V is infinite dim'l.

i.e V is infinite dim'l if V is not finite dim'l.

Ex: P = polynomial function from $\mathbb{R} \to \mathbb{R}$ is infinite dim'l

Proof: Sps $\mathbb{R}^m \xrightarrow{\varphi} P$ is isom.

Thus φ is onto, and hence any polynomial is a lin. combo of the polynomials $\varphi(e_1), \ldots, \varphi(e_m)$.

Let $n = \max\{\deg(\varphi(e_1)) \ldots \deg \varphi(e_m)\}$

Clearly any lin combo of $\varphi(e_1) \ldots, \varphi(e_m)$ has degree at most n, contradicting onto property of φ.

<u>Matrix</u> of a <u>lin transf</u> $T: V_1 \longrightarrow V_2$

Assume V_1, V_2 are finite dim'l $\dim(V_1) = n$, $\dim(V_2) = m$

Let $\varphi: \mathbb{R}^n \longrightarrow V_1$ be an isom, $\psi: \mathbb{R}^m \to V_2$ isom.

Basis of $V_1 = \{f_1, \ldots, f_n\} = \{\varphi(e_1) \ldots \varphi(e_n)\}$
Basis of $V_2 = \{g_1, \ldots g_m\} = \{\psi(e_1), \ldots, \psi(e_m)\}$

Note
$$\begin{tikzcd} V_1 \arrow[r, "T"] & V_2 \arrow[d, "\psi^{-1}"] \\ \mathbb{R}^n \arrow[u, "\varphi"] \arrow[r, dashed, "\psi^{-1} \circ T \circ \varphi"'] & \mathbb{R}^m \end{tikzcd}$$

$\psi^{-1} \circ T \circ \varphi: \mathbb{R}^n \to \mathbb{R}^m$ is a lin. transf hence given by $\vec{x} \mapsto A_{m \times n} \vec{x}$.

We say A is the matrix of T wrt basis $\{f_1 \ldots f_n\}$ of V_1 and $\{g_1 \ldots g_m\}$ of V_2.

<u>Claim</u>: j-th col of A equals $\begin{pmatrix} c_1 \\ c_m \end{pmatrix}$ where
$$T(f_j) = c_1 g_1 + \ldots + c_m g_m.$$

<u>Pf</u>: Sps $T(f_j) = c_1 g_1 + \ldots + c_m g_m$. Then
$$(\psi^{-1} \circ T \circ \varphi)(e_j) = \begin{pmatrix} c_1 \\ c_m \end{pmatrix}$$ but $\psi^{-1} \circ T \circ \varphi = A$ ∎

Example: $T: P_m \to P_m$ $\quad f \mapsto f'$

Basis: $B = \{1, x, \dots, x^m\}$ 1) Clearly B spans P_m

2) If $\sum_{i=0}^{m} c_i x^i = p(x) = 0$ (function) then it has more than m roots $\Rightarrow p \equiv 0$ i.e. all $c_i = 0$

So $1, x, \dots, x^m$ are lin. indep.

We can also use basis $B' = \{1, x, \frac{x^2}{2!}, \dots, \frac{x^m}{m!}\}$

matrix of T wrt B is A then j-th col of A

= coords of $T(x^j) = jx^{j-1}$ wrt $\{1, x, \dots, x^m\}$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & 0 & & \ddots & m \\ & & & & 0 \end{bmatrix}$$

matrix of T wrt basis B' is
$$\begin{bmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & 0 & & \ddots & 1 \\ & & & & 0 \end{bmatrix}.$$

$T: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad$ basis $\vec{u}, \vec{v}, \vec{u} \times \vec{v}$ where $\|u\| = \|v\| = 1$
$x \mapsto \vec{u} \times \vec{x}$ $\qquad \vec{u} \cdot \vec{v} = 0$.

matrix is
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Ex: $V = \operatorname{Span}\{\cos x, \sin x\}$ in $C^\infty(\mathbb{R}; \mathbb{R})$

$T: V \to V$ $\qquad$ 1) is T isom
$f \mapsto 3f + 2f' - f''$ $\qquad$ 2) matrix of T wrt basis $\cos x, \sin x$

A: 2) $\begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix}$ $\quad$ $\det = 16 + 4 = 20 \neq 0$ So invertible.

E.g. $T: M_{2\times 2} \longrightarrow M_{2\times 2}$

$$M \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} M - M \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

a) matrix of T wrt std-basis of $M_{2\times 2}$

b) Bases for $\operatorname{img} T$, $\ker T$

A. a)
$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

b) $\operatorname{Im}(T)$: Basis
$$\begin{matrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{matrix}$$

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

c) $\operatorname{Ker}(T)$ Basis: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

<u>Lemma</u>: If $T: V_1 \longrightarrow V_2$ is isom, V_1, V_2 finite dim'l. and $W \subset V_1$ is subspc then $\dim(T(W)) = \dim(W)$

<u>Pf</u>. Pick basis $B = \{w_1 \dots w_m\}$ of W. We claim $\{T(w_1) \dots T(w_m)\} = B'$ is a basis of $T(W)$. Clearly B' spans $T(W)$.

If $\sum c_i T(w_i) = 0 \iff T(\sum c_i w_i) = 0$

$$\Downarrow$$

$$\sum c_i w_i = 0$$

$$\Downarrow$$

all $c_i = 0$. ✓.

Rank Nullity Thm:

Q4 in Assignment 6:

Q6 in Assignment 6.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T"] & \mathbb{R}^m \\ \arrow[u, "\varphi"] & \end{tikzcd}$$

$$V \xrightarrow{T} W$$

$\dim V = n$, $\dim W = m$

Pick basis $\{f_1 \dots f_n\}$ of V, Basis $\{g_1 \dots g_m\}$ of W

So $\varphi : \mathbb{R}^n \longrightarrow V$ and $\psi : \mathbb{R}^m \longrightarrow W$ are ISOMs.

$\begin{pmatrix} c_1 \\ c_n \end{pmatrix} \mapsto \Sigma c_i f_i$ $\qquad$ $\begin{pmatrix} d_1 \\ d_m \end{pmatrix} \mapsto \Sigma d_i g_i$

$\mathbb{R}^n \xrightarrow{\psi^{-1} T \varphi} \mathbb{R}^m$ is $\vec{x} \mapsto \underset{m\times n}{A} x$.

$$\text{Im}(A) = \text{Im}(\psi^{-1} T \varphi) = \psi^{-1} \text{Im}(T\varphi) = \psi^{-1}(\text{Im}\, T)$$

Thus $\dim(\text{Im}\, A) = \text{rank}(A) = \dim(\psi^{-1}(\text{Im}\, T)) = \dim(\text{Im}\, T) = \text{rank}(T)$

$$\text{Ker}(A) = \text{Ker}(\psi^{-1} T \varphi) = \text{Ker}(T\varphi) = \varphi^{-1}(\text{Ker}\, T)$$

$\Rightarrow$ $\text{nullity}(A) = \dim(\text{Ker}\, A) = \dim(\varphi^{-1}(\text{Ker}\, T)) = \dim(\text{Ker}\, T) = \text{nullity}(T)$

Thus $\underline{\text{nullity}(T) + \text{rank}(T) = \text{nullity}(A) + \text{rank}(A) = n = \dim V}$

'RANK-NULLITY theorem'