

Sept 14 - 2015

Recall: A function $T: V \rightarrow W$ between V & W is said to be a lin. trans if $T(av_1 + bv_2) = aT(v_1) + bT(v_2)$ for all $a, b \in \mathbb{R}$, $v_1, v_2 \in V$.

We say V is finite dim'd if $\exists$ a lin trans.

$\rightarrow \varphi: \mathbb{R}^m \rightarrow V$ which is bijective (bijective
lin trans also
called ISOMORPHISM)

In this case we define $\{\varphi(e_1), \dots, \varphi(e_m)\}$ to be a basis of V , and $m := \dim(V)$.

Note: if $\psi: \mathbb{R}^n \rightarrow V$ is another isomorphism, then

$\psi^{-1} \circ \varphi: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is isom. (bijective lin trans)
hence $m = n$.

Thus $\dim(V)$ is well defined.

Def: If no such $\varphi: \mathbb{R}^m \rightarrow V$ (isomorphism) exists for any $m \in \mathbb{N}$, we say V is infinite dim'd.

i.e. V is infinite dim'd if V is not finite dim'd.

Ex: $\mathcal{P}$ = polynomial function from $\mathbb{R} \rightarrow \mathbb{R}$ is infinite dim'd.

Proof: Skp $\mathbb{R}^m \xrightarrow{\varphi} \mathcal{P}$ is isom.

Thus φ is onto, and hence any polynomial is a lin. combo of the polynomials $\varphi(e_1), \dots, \varphi(e_m)$.

Let $n = \max\{\deg(\varphi(e_1)) - \dots - \deg \varphi(e_m)\}$ 2)

Clearly any lin combo of $\varphi(e_1), \dots, \varphi(e_m)$ has degree at most n , contradicting onto property of φ .

Matrix of a lin transt $T: V_1 \longrightarrow V_2$

Assume V_1, V_2 are finite dim'd $\dim(V_1) = n, \dim(V_2) = m$

Let $\varphi: \mathbb{R}^n \rightarrow V_1$ be an isom, $\psi: \mathbb{R}^m \rightarrow V_2$ isom.

Basis of $V_1 = \{f_1, \dots, f_n\} = \{\varphi(e_1), \dots, \varphi(e_n)\}$

Basis of $V_2 = \{g_1, \dots, g_m\} = \{\psi(e_1), \dots, \psi(e_m)\}$

$$\begin{array}{ccc} \text{Note } V_1 & \xrightarrow{T} & V_2 \\ \uparrow \varphi & & \downarrow \psi^{-1} \\ \mathbb{R}^n & \xrightarrow{\psi^{-1} \circ T \circ \varphi} & \mathbb{R}^m \end{array}$$

$\psi^{-1} \circ T \circ \varphi: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a lin. transt hence given by $\vec{x} \mapsto A \vec{x}$ where A is $m \times n$.

We say A is the matrix of T w.r.t basis

$\{f_1, \dots, f_n\}$ of V_1 and $\{g_1, \dots, g_m\}$ of V_2 .

Claim: j -th col of A equals $\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$ where

$$T(f_j) = c_1 g_1 + \dots + c_m g_m.$$

Pf: Skz $T(f_j) = c_1 g_1 + \dots + c_m g_m$. Then

$$(\psi^{-1} \circ T \circ \varphi)(e_j) = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \text{ but } \psi^{-1} \circ T \circ \varphi = A$$



Example: $T: P_m \rightarrow P_m \quad f \mapsto f'$ (3)

Basis: $B = \{1, x, \dots, x^m\}$ 1) Clearly B spans P_m

2) If $\sum_{l=0}^m c_l x^l = p(x) = 0$ (function) then it has more than m roots

So $1, x, \dots, x^m$ are lin-indep. $\Rightarrow p \equiv 0$ i.e. all $c_l = 0$

We can also use basis $B' = \{1, x, \frac{x^2}{2!}, \dots, \frac{x^m}{m!}\}$

matrix of T wrt B is A then j -th col of A

= coords of $T(x^j) = jx^{j-1}$ wrt $\{1, x, \dots, x^m\}$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & m \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

matrix of T wrt basis B' is $\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$

$T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ basis $\vec{u}, \vec{v}, \vec{u} \times \vec{v}$ where $\|\vec{u}\| = \|\vec{v}\| = 1$
 $\vec{u} \cdot \vec{v} = 0$
 $\vec{u} \mapsto \vec{u} \times \vec{u} = \vec{0}$

matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

Ex: $V = \text{Span}\{\cos x, \sin x\}$ in $C^\infty(\mathbb{R}; \mathbb{R})$

$T: V \rightarrow V$

1) Is T isom

$f \mapsto 3f + 2f' - f''$

2) matrix of T wrt basis $\cos x, \sin x$

A: 2) $\begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix}$ $\det = 16 + 4 = 20 \neq 0$ So invertible.

$$E_1 = T: M_{2 \times 2} \longrightarrow M_{2 \times 2} \quad 4)$$

$$M \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} M - M \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

a) matrix of T wrt std-basis of $M_{2 \times 2}$

b) Bases for $\text{Im } T$, $\text{Ker } T$

A. a)

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

b) $\text{Im}(T)$: Basis

$$\begin{array}{c} 0 \quad 1 \\ \swarrow \quad \searrow \\ 1 \quad 0 \\ \swarrow \quad \searrow \\ 0 \quad 0 \\ \swarrow \quad \searrow \\ 0 \quad 1 \end{array}$$

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\Rightarrow \text{Ker}(T)$ Basis: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Lemma: If $T: V_1 \longrightarrow V_2$ is isom, V_1, V_2 finite dim'd.

and $W \subset V_1$ is subspace then $\dim(T(W)) = \dim(W)$

Pf. Pick basis $B = \{w_1, \dots, w_m\}$ of W . We claim $\{T(w_1), \dots, T(w_m)\} = B'$

is a basis of $T(W)$. Clearly B' spans $T(W)$.

$$\text{If } \sum c_i T(w_i) = 0 \iff T(\sum c_i w_i) = 0$$

$\Downarrow$

$$\sum c_i w_i = 0$$

$\Downarrow$

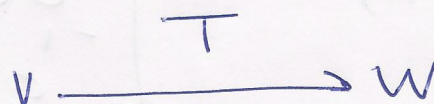
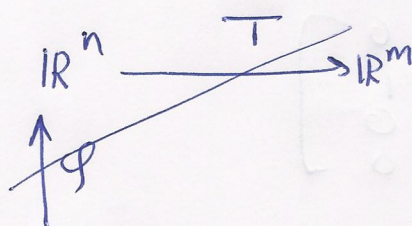
$$\text{all } c_i = 0. \checkmark$$

Rank Nullity Thm:

(5)

Q4 in Assignment 6:

Q6 in Assignment 6:



$$\dim V = n, \quad \dim W = m$$

Pick basis $\{f_1, \dots, f_n\}$ of V , Basis $\{g_1, \dots, g_m\}$ of W

So $\varphi: \mathbb{R}^n \rightarrow V$ and $\psi: \mathbb{R}^m \rightarrow W$ are ISOMs.
 $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \mapsto \sum c_i f_i$ and $\begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix} \mapsto \sum d_i g_i$

$$\mathbb{R}^n \xrightarrow{\bar{\psi}^T T \varphi} \mathbb{R}^m \mapsto \vec{x} \mapsto A \vec{x}_{m \times n}$$

$$\text{Im}(A) = \text{Im}(\bar{\psi}^T T \varphi) = \bar{\psi}^T \text{Im}(T \varphi) = \bar{\psi}^T (\text{Im } T)$$

$$\text{Thus } \dim(\text{Im } A) = \text{rank}(A) = \dim(\bar{\psi}^T (\text{Im } T)) = \dim(\text{Im } T) = \text{rank}(T)$$

$$\text{Ker}(A) = \text{Ker}(\bar{\psi}^T T \varphi) = \text{Ker}(T \varphi) = \bar{\psi}^T (\text{Ker } T)$$

$$\Rightarrow \text{nullity}(A) = \dim(\text{Ker } A) = \dim(\bar{\psi}^T (\text{Ker } T)) = \dim(\text{Ker } T) = \text{nullity}(T)$$

$$\text{Thus } \text{nullity}(T) + \text{rank}(T) = \text{nullity}(A) + \text{rank}(A) = n = \dim V$$

RANK-NULLITY theorem