Week 6 Sep 7, Sep 8: Vector Spaces and Linear Transf.
(Chapter 4 of textbook)

Week 7: Sept 14 : Start Chap 5
Sept 15 : Review for Mid term exam.

Week 8: Sep 22: Mid Term exam 3-5pm
in LHC 101, 103, 201.

Week 9 : nothing

Week 10 : Oct 5 : Dr. Dipramit takes over.

Chapter 4.

given subspc $W \subseteq \mathbb{R}^n$, we saw there exists a lin transf

$T_A : \mathbb{R}^m \to \mathbb{R}^n$ such that $\begin{array}{l} \mathbb{R}^m \longrightarrow W \\ \vec{x} \longmapsto A\vec{x} \end{array}$ is bijective (and linear)

So although W is not $\mathbb{R}^m$, it behaves like it is, or for practical purposes it can be thought of as $\mathbb{R}^m$.

A vector spc informally is a set which behaves like $\mathbb{R}^m$ (scalar multiplicn, linear combinations etc).

Example of vector spc.

Let P_m = set of polynomials with real coeff's of degree at most $m-1$

$$= \{a_0 + a_1 x + \cdots + a_{m-1} x^{m-1} : \quad a_0, \ldots, a_{m-1} \in \mathbb{R}\}$$

Clearly given $p(x) = \sum_{i=0}^{m-1} a_i x^i$ and $q(x) = \sum_{i=0}^{m-1} b_i x^i$

we can take lin. combo. $c\,p(x) + d\,q(x) = \sum_{i=0}^{m-1} (ca_i + db_i) x^i$

where $c, d \in \mathbb{R}$.

If we consider: $T : \mathbb{R}^m \longrightarrow P_m$

$$\vec{\begin{pmatrix} a_0 \\ \vdots \\ a_{m-1} \end{pmatrix}} \mapsto \sum_{i=1}^{m} a_i x^i$$

then T is a bijection. Moreover if $T(\vec{a}) = p$, $T(\vec{b}) = q$

then $c\,p(x) + d\,q(x)$ as defined above is $T(c\vec{a} + d\vec{b})$

Thus via T, we see P_m although not equal to $\mathbb{R}^m$ is practically like $\mathbb{R}^m$.

Infact we can go ahead and see $\{x^0, x^1, \ldots, x^{m-1}\}$ are like a basis for P_m.

<u>Def^n</u>. A vector <u>space</u> or <u>linear space</u> V is a set together with a) an addition operation $v, u \mapsto v+u$
b) a scalar multiplication $c, v \mapsto cv$

satisfying some conditions:

1) $u+v = v+u$, $(u+v)+w = u+(v+w)$

2) for $c, d \in \mathbb{R}$, $u, v \in V$

$$(c+d)\,u = cu + du$$
$$(cd)\,u = c(du)$$
$$c(u+v) = cu + cv$$

3) There exists an "Additive identity" O_V in V satisfying.

$u + O_V = u$ for all $u \in V$.

4) $0 \cdot u = O_V \quad \forall u \in V$
$1 \cdot u = u \quad \forall u \in V$

<u>Note</u>: 1) O_V is unique <u>Pf</u>: Sps O'_V is another Additive identity: $O'_V + u = u \; \forall u \Rightarrow O'_V + O_V = O_V$
But $O'_V + O_V = O_V + O'_V = O'_V$ hence $O_V = O'_V$.

2) $1 \cdot u + -1 \cdot u = (1+-1)\cdot u = 0 \cdot u = O_V$
Hence every element of V has an additive inverse. We will denote $-1 \cdot u$ by $-u$.
We will also drop the subscript V on O_V and just denote it by O. There is no confusion between $0 \in \mathbb{R}$, $O \in V$. Besides $0 \cdot v = O$.

Examples of Vector spaces

1) Let S be any set. Let V_S = Set of all functions from $S \to \mathbb{R}$.

If $f, g, h \in V$, $c, d \in \mathbb{R}$ then

we can define $f+g$ to be the function s.t

$(f+g)(s) = f(s) + g(s)$

We can define cf to be the function: $(cf)(s) = cf(s)$.

Clearly V satisfies all properties 1–4 required.

0_{V_S} = the function f satisfying $f(s) = 0 \ \forall s \in S$.

Q: If $S = \{1\}$ what is V_S. Ans $\mathbb{R}$

Q: If $S = \{1, 2, .., n\}$ what is V_S. Ans $\mathbb{R}^n$.

Q: If $S = \mathbb{N} = \{1, 2, 3 ----\}$ what is V_S:

Ans. vector space of all real sequences.

__Def__: If W is a subset of a v.s V, we say W is a subspace if whenever $w_1, w_2 \in W$ and $a, b \in \mathbb{R}$ the lin. combo. $aw_1 + bw_2$ also is in W.

(W is closed w.r.t lin. combinations).

(Note Take $a = b = 0$ to get $0_V \in W$)

Q: What is the "Smallest Subspace" of V_S, $S = \mathbb{N}$ which contains all the $V_{\{1,2,..n\}}$ for all n.

A: It is the vector space of "eventually zero" real sequences.
__Proof__: Tutorial.

Remark: If W is a subspace of V. Then forgetting about V, W is a v.s. in its own right as it satisfies conditions 1–4.

Another example: Take $V = V_{\mathbb{R}}$ = all functions from $\mathbb{R}$ to $\mathbb{R}$.

$V \supset C(\mathbb{R},\mathbb{R}) = \{$cts functions from $\mathbb{R} \to \mathbb{R}\} \supset C^{\infty}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}$ s.t $f', f'' \ldots$ all exist$\}$

$C^{\infty}(\mathbb{R}) \supset P$ = all polynomials with real coeffs $\supset P_m$ = polynomials of degree $\leq m$ with real coeffs.

Another example: $V = Mat_{m,n}$ = space of all $m \times n$ real matrices.

$O_V = O_{m \times n}$ matrix, $A, B \in V$, $d, c \in \mathbb{R}$ it is clear what is $cA + dB$.

Def^n: A set $\{v_1, v_2 \ldots v_n\} \subseteq V$ is said to

1) span V if every $v \in V$ is a lin combo of $v_1 \ldots v_n$

2) be l.i if $\sum_{i=1}^{n} c_i v_i = O_V \Rightarrow$ all $c_i = 0$.

equivalently if $T : \mathbb{R}^m \to V$ is the function.

$$\begin{pmatrix} a_1 \\ a_m \end{pmatrix} \mapsto \sum_{i=1}^{m} a_i v_i$$

then 1') T is onto

2') T is 1-1

We say $\{v_1 - v_n\}$ is a basis of V if both 1) and 2) are satisfied. In other words T is bijective.

Def: A functn $T: V \to W$ between two V.S V and W is said to be a LIN. TRANSF. if

$$T(av_1 + bv_2) = aT(v_1) + bT(v_2) \quad \forall\ v_1, v_2 \in V$$
$$a, b \in \mathbb{R}.$$

Def: We say $T: V \to W$ a lin transf is an ISOMORPHISM if T is bijective (both 1-1 and onto).

Example of Basis:

If $V = P_m$ then $\{1, x, x^2 \ldots x^m\}$ is a Basis.

Pf: • clearly Span of this set is all of P_m

• If $f = \sum_{i=0}^{m} a_i x^i = 0$, function. then f has more than m roots

$\Rightarrow$ ~~$f = 0$~~. hence all a_i are zero.

Also $\{1, x, x^2/2, x^3/6, \ldots, x^m/m!\}$ is a basis.

Thm: Any two bases of V have same # of elements