MTH 201, Autumn 2015, Week 4 Aug 24

1) Finish loose ends from Chapter 2
2) Cover Chapter 3

1) Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be non-zero vectors.
We define $\varphi_{\vec{u}} : \mathbb{R}^n \to \mathbb{R}$, $\vec{x} \mapsto u^T x = \vec{u} \cdot \vec{x}$ and $\psi_v : \mathbb{R} \to \mathbb{R}^n$, $t \mapsto t\vec{v}$

These are lin. transf. as they are given by matrices u^t and V resp.

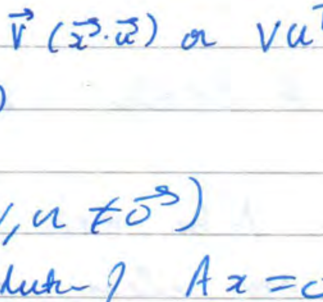
The lin transf. $\psi_v \circ \varphi_u : \mathbb{R}^n \to \mathbb{R}^n$ take $\vec{x}$ to $\vec{v}(\vec{x} \cdot \vec{u})$ or $Vu^T x$
Thus its matrix is Vu^t (which is $n \times n$)

Q: What is Rank of matrix Vu^t (where $V, u \neq \vec{0}$)
Hint (rk of matrix $A_{m \times n}$ is # d.o.f of solution of $Ax = 0$)

If $v = u$ we called $\psi_v \circ \varphi_u$ as T_u in HW3 (when $\vec{u} \cdot \vec{u} = 1$ additionally)

Basically $\vec{x} \mapsto \dfrac{(\vec{u} \cdot \vec{x})}{\vec{u} \cdot \vec{u}} \vec{u}$ is projection of $\vec{x}$ on direction of $\vec{u}$

when $\vec{u} \cdot \vec{u} = u^t u = 1$ projection is T_u.



$$x = T_u x + x - T_u x$$
$$= x_{\parallel} + x_{\perp}$$
(where $x_{\parallel}$ is parallel to $\vec{u}$, $x_{\perp}$ is $\perp$ to $\vec{u}$)

Clearly $x_{\perp} - x_{\parallel} = x - 2x_{\parallel} = (\mathbb{1} - 2T_u)x$ is reflection of $\vec{x}$ in plane $\perp$ to $\vec{u}$.

Proposition

If A is a $n\times n$ Sq matrix and $\exists$ a $n\times n$ matrix B s.t one of $AB = I_n$; $BA = I_n$ then $B = A^{-1}$

(original def^n requires both these conditions)

Proof: If $AB = I_n$ then $ABx = x$ so $ABx=0$ only if $x=0$
Thus $Bx=0$ only if $x=0$ $\Rightarrow$ B is invertible

Multiply $AB=I_n$ by B^{-1} on right to get $A = B^{-1}$. Hence $A^{-1}=B$.

If $BA=I_n$ above shows $B=A^{-1}$.

Q: Let A be $m\times n$ matrices What does $A\mapsto AE$ do where E is $n\times n$ elem matrix?

A: If E interchanges i-th and j-th row then $E=E^t$ so $A^t \mapsto E^tA^t = EA^t$ interchanges i-th and j-th row of A^t. Thus ~~AE~~ $A\mapsto AE$ interchanges i-th j-th cols.

If E rescales i-th row by c, then $E=E^t$ so $\cancel{E^tA^t = EA^t}$
$A^t \mapsto E^tA^t = EA^t$ rescales i-th row of A^t
$A\mapsto AE$ rescales i-th col of A by c

If ~~$E\mapsto EA$ adds~~ E corresponds to adding $c\times j$-th row to i-th row
the $A^t\mapsto E^tA^t$ changes $\underline{j\text{-th row}}$ of A^t to j-th row + $c\cdot i$-th row
Thus $A\mapsto AE$ changes i-th col of A to i-th col + c j-th column.

3 definitions.

1) A subset $W \subset \mathbb{R}^n$ is called a subspace if whenever $w_1, w_2 \in W$ so does $c_1 w_1 + c_2 w_2$ $\forall$ $c_1, c_2 \in \mathbb{R}$.

(informally a subset is a Subspace if it is closed wrt. Taking linear combinations)

Note taking $w_1 = w_2$ and $c_1 = c_2 = 0$ we see $\vec{0} \in W$

Ex: 1) $\mathbb{R}^k \subset \mathbb{R}^n$ sitting as $\{(x_1, \ldots x_k, 0 \ldots 0)^t : x_i \in \mathbb{R}\}$

2) Let $\vec{v} \in \mathbb{R}^n$ and consider $W = \{c\vec{v} : c \in \mathbb{R}\}$

3) Let $v_1, \ldots, v_m \in \mathbb{R}^n$ consider $W = \{c_1\vec{v_1} + \ldots + c_m\vec{v_m} : c_i \in \mathbb{R}\}$

2) If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a lin transf we define

$Im(T) = \{Tx : x \in \mathbb{R}^m\} \subset \mathbb{R}^n$ "image of T"

$Ker T = \{x \in \mathbb{R}^m : Tx = 0\} \subset \mathbb{R}^m$ "Kernel of T".

Claim: $Im T$ is subspace of $\mathbb{R}^n$, $Ker T$ is subspace of $\mathbb{R}^m$

Pf: • If $w_1, w_2 \in Im T$, write $w_1 = Tv_1$, $w_2 = Tv_2$, then $c_1\vec{w_1} + c_2\vec{w_2} = c_1 Tv_1 + c_2 Tv_2 = T(c_1 v_1 + c_2 v_2) \in Im T$.

• If $Tv_1 = 0$, $Tv_2 = 0$ then $T(c_1\vec{v_1} + c_2\vec{v_2}) = c_1 Tv_1 + c_2 Tv_2 = \vec{0} + \vec{0} = \vec{0}$.

Q: What are Subsp of $\mathbb{R}$?

A: We certainly have $\{0\}$, $\mathbb{R}$. If W is a subsp, $W \neq \{0\}$ pick a vector $\vec{a} \in W$, $\vec{a} \neq \vec{0}$ then clearly $\{c\vec{a} : a \in \mathbb{R}\} \subset W$
$= \mathbb{R}$.

So $\{0\}$, $\mathbb{R}$ only Subspc. of $\mathbb{R}$

Q: What are Subsp. of $\mathbb{R}^2$?

A: $\{0\}$, $\{\mathbb{R}^2\}$, $\{t\vec{v} : t \in \mathbb{R}\}$ $\vec{v} \neq 0$ are Subspcs. We claim there are no others.

Let $W \neq \{0\}$ be Subsp of $\mathbb{R}^2$. Pick $\vec{v} \in \vec{W}$, if $\vec{W} \neq \{c\vec{v} : c \in \mathbb{R}\}$

$\exists\ \vec{u} \in \vec{W}$ s.t. $\vec{u} \neq c\vec{v}$ for any $c \in \mathbb{R}$.

Consider $A = [\vec{u}, \vec{v}]_{2\times 2}$, $A\vec{x} = \vec{0}$ has only $\vec{x} = 0$ as Sol^n
why?

So for every $\vec{b} \in \mathbb{R}^2$ $A\vec{x} = \vec{b}$ has unique Sol^n $(x = A^{-1}b)$
In other words $\mathrm{Im}(T) = \mathbb{R}^2$. But $\mathrm{Im}\,T \subset W$, so $W = \mathbb{R}^2$

Q: Consider $W = \{(x_1, x_2, x_3)^t \in \mathbb{R}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

- Find T_A s.t. $\mathrm{Im}(T_A) = W$
- Find matrix B s.t. $\mathrm{Ker}(T_B) = W$

A: Any $3 \times m$ matrix, whose cols ~~are $\perp$ to~~ Span $\begin{pmatrix}1\\2\\3\end{pmatrix}^{\perp}$

eg: $\begin{pmatrix}-2 & 0\\ 1 & -3\\ 0 & 2\end{pmatrix}$ $m=2$ works.

$B = [1 \ \ 2 \ \ 3]$

Lemma: If $W \subset \mathbb{R}^n$ subspc, $\exists$ $n \times m$ matrix A with $m \leq n$

such that $\operatorname{Im}(T_A) = W$ and T_A is 1-1.

Pf: We will come up with a sequence of matrices $A(1), A(2) \cdots A(m)$ where $A(i)$ has size $n \times i$. W will be $\operatorname{Im}(T_{A(m)})$

Let $A(1) = \begin{bmatrix} v_1 \end{bmatrix}_{n \times 1}$ where v_1 is any non zero vector. If $\operatorname{Im}(T_{A(1)}) = W$ stop.

Let $A(2) = \begin{bmatrix} v_1 \mid v_2 \end{bmatrix}_{n \times 2}$ where $v_2 \neq c v_1$ for any $c \in \mathbb{R}$, $v_2 \in W$

If $\operatorname{Im}(T_{A(2)}) = W$ stop.

$\vdots$

Let $A(i+1) = [v_1 | \cdots | v_i | v_{i+1}]$ where

v_{i+1} is not a lin. comb of $v_1 \cdots v_i$, $(v_{i+1} \in W)$

This process must terminate at some stage $m \leq n$

why?: because if we are at stage i it means

$A(i)\vec{x} = \vec{0}$ has only $\vec{0}$ soln i.e. $\operatorname{Rank}(A(i)) = i$ ~~$\leq \min\{n, i\}$~~

But $\operatorname{Rank}(A(i)) \leq \min\{n, i\}$. $\Rightarrow$ $i \leq n$. ∎

(T_A 1-1 $\Leftrightarrow$ $\operatorname{Rank}(A) = \#$ cols of A. At stage m $\operatorname{Im} T_A = W$; T_A is 1-1)

Defn: given subspc $W \subset \mathbb{R}^n$, we define $\dim(W) = m$. where m is as in Lemma above.

Q: Can we have $\dim(W) = m$, $\dim(W) = p$, $m \neq p$?

A: No. Sps $T: \mathbb{R}^m \to \mathbb{R}^n$ both are 1-1 and have
$$\begin{array}{c} \uparrow S \\ \mathbb{R}^p \end{array} \quad \text{Image} \to W$$

If $f: \mathbb{R}^m \to W$ is the (another name for) function defined by T then by hypothesis f is bijective so $f^{-1}: W \to \mathbb{R}^m$ exists

We note that $f^{-1}: W \to \mathbb{R}^m$ satisfies $f^{-1}(aw_1+bw_2) = af^{-1}(w_1)+bf^{-1}(w_2)$
Why? pick $V_1, V_2 \in \mathbb{R}^m$ s.t $f(V_i) = W_i$, $i=1..2$, in other words $V_i = f^{-1}(W_i)$

then $af^{-1}(w_1)+bf^{-1}(w_2) = aV_1+bV_2$. Hence ~~$f(aV_1+bV_2)=$~~
$f(aV_1+bV_2) = T(aV_1+bV_2) = aTV_1 + bTV_2 = af(V_1)+bf(V_2) = aw_1+bw_2$
Thus $f^{-1}(aw_1+bw_2) = aV_1+bV_2 = af^{-1}(w_1)+bf^{-1}(w_2)$

Similarly if $g: \mathbb{R}^p \to W$ denotes the function from $\mathbb{R}^p \to W$ determined by S.
then $g^{-1}: W \to \mathbb{R}^p$ satisfies linearity.

Thus $\mathbb{R}^p \xrightarrow{f\circ g^{-1}} \mathbb{R}^m$ and $\xrightarrow[\mathbb{R}]{g\circ f^{-1}}$

$\mathbb{R}^m \xrightarrow{g^{-1}\circ f} \mathbb{R}^p$ is a lin. transf which is bijective
(its inverse is $f^{-1}\circ g$)

If $C_{p\times m}$ is matrix of $g^{-1}\circ f$ it is invertible, hence $p=m$ ∎

Note: A basis for a subspc $W \subseteq \mathbb{R}^n$ is a collection of vectors $\{w_1, w_2, \ldots, w_m\}$ s.t every element of W is a lin. combo of $w_1, \ldots, w_m$, in a unique way.
(This is same as saying that T_A is bijective onto W where A is the $n\times m$ matrix with columns $w_1, w_2 \ldots w_m$)

Rank- Nullity theorem.

Lemma: 1) If $A = \left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right]_{m\times n}$ then $rank(A) = r$ ~~nullity(A)=~~

$\dim(\ker A) = \dim\{e_{r+1}, \ldots, e_n\} = \text{Img of } \mathbb{R}^{n-r} \longrightarrow \mathbb{R}^n$, $\vec{x} \mapsto (\vec{0}, \vec{x})$ is 1-1

Hence $\dim \ker(A) = n - r$.

Thus calling $nullity(A) = \dim(\ker A)$ we note

$$\boxed{nullity(A) + Rank(A) = n}$$

Thm: If A is $m\times n$; and $Q_{m\times m}$ and $P_{n\times n}$ are invertible matrices

then $nullity(A) = nullity(QAP)$, $rank(A) = rank(QAP)$

Proof: Lemma: 2) Let $W \subset \mathbb{R}^n$ and $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be lin trans with A invertible then $\dim(W) = \dim(T_A W)$

Pf: If $w_1 \ldots w_r$ is basis for W, then $Tw_1, \ldots, Tw_r$ spans $T_A W$ and if $\sum c_i Tw_i = 0$ then $T(\sum c_i w_i) = 0$. But T is invertible so $\sum c_i w_i = 0 \Rightarrow c_i = 0 \ \forall i$ because $w_1 \ldots w_n$ are l.i.

$\ker(QAP) = \{x : QAPx = 0\} = Q.\ker(AP)$. By Lemma 2, $nullity(QAP) = nullity(AP)$

So $\ker(AP) = P^{-1}\ker(A)$ So by lemma 2 again $nullity(AP) = nullity\ A$

Thus we have shown $nullity(A) = nullity(QAP)$

$rk(QAP) = \dim(Q . \text{Im}(AP)) \xrightarrow{\text{lemma 2}} \dim(\text{Im}(AP)) = \dim(\text{Im} A) = Rank(A)$

becau $AP . \mathbb{R}^n = A\mathbb{R}^n$

Let A be an $m \times n$ Matrix. We have seen that there exists an invertible $m \times m$ matrix $Q = E_s E_{s-1} \cdots E_1$ s.t QA is RREF.. Let $P = E_1' E_2' \cdots E_t'$ be a sequence of $n \times n$ elem matrices then by Thm above $rk(A) = rk(QAP)$ and $nullity(A) = nullity(QAP)$

Q How to Pick P wisely so that $\underbrace{\cancel{QAP}}_{MP}$ is as simple as possible where M = RREF $m \times n$ matrix

A: We can pick P s.t $M \cdot P = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$ where $r = rk(M)$

Subtract ~~p~~ suitable multiple of pivot Colums from non pivot Colums. Then for $i = 1 .. r$ the i-th row looks like $(e_{p_i})^T$. Now use Column interchanges to bring all pivot Colums to the left, in the form $\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$.

Thus $rk(A) = rk(QAP) = rk\left(\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]\right) = r$

$null(A) = nullity(QAP) = nullity\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right) = n - r$

Thus $rk(A) + nullity(A) = n$ for $m \times n$ matrix A

RANK - NULLITY THEOREM

Another defn: $v_1, v_2 .., v_k \in \mathbb{R}^n$ are called lin. indep if matrix $A_{n \times k} = [v_1\ v_2\ \cdots\ v_k]$ has rank k $\iff (Ax = 0 \Rightarrow x = 0)$

$\iff \left(\sum_{i=1}^{k} c_i \vec{v_i} = 0 \Rightarrow \text{all } c_i = 0\right)$